# On inflationary perturbations

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# Abstract

In this work we consider different aspects of primordial perturbations generated during inflation. In the first part we discuss the production of *curvature perturbations*. The standard single-field inflation model predicts almost scale-invariant adiabatic perturbations which obey Gaussian statistics. This prediction is in very good agreement with the present observational data. However the measure of the level of non-Gaussianity is not precise yet and a deviation from Gaussianity is allowed. The curvaton scenario, a model of inflation with two scalar fields, may produce a higher level of non-Gaussianity of the perturbations than the single-field model. In this thesis we discuss the *super-curvaton scenario*, a curvaton model which naturally appears in the context of the simplest model of chaotic inflation in supergravity. We compute the non-linearity parameter $f_{NL}$ and show that the level of non-Gaussianity can be in the observationally interesting range from $\mathcal{O}(10)$ to $\mathcal{O}(100)$.
In the second part we discuss the generation of *large-scale magnetic fields* from the amplification of quantum fluctuations during inflation. We consider a very broad class of models that can break the conformal invariance of electromagnetism and therefore give rise to long-wave magnetic fields. We study the effect of the *back reaction* of the generated field on the background and show that they can be very important. Assuming that the back reaction does not spoil inflation and requiring that inflation lasts at least 75 e-folds, we find a rather strong restriction on the amplitude of the primordial fields which could be generated on inflation. Namely, this amplitude cannot exceed $10^{-32}$ G on Mpc scales today. This magnetic field is too small to explain the field observed in the Universe and it is too weak also to be amplified to the observable values by the galactic dynamo mechanism.

# Contents

# Chapter 1

# Introduction

According to our current understanding of the Universe, all cosmological structures that we see today in the sky, like galaxies, clusters of galaxies and all the web of the large-scale structure, and the observed temperature anisotropies in the Cosmic Microwave Background (CMB), have a *quantum* origin.

The mechanism responsible for these processes is called *cosmological inflation*, an epoch of accelerated expansion of the Universe that took place $10^{-35}$ seconds after the Big Bang. Inflation predicts the observable Universe to be spatially flat, homogeneous and isotropic on large angular scales, and the primordial inhomogeneity to be almost scale-invariant and to obey Gaussian statistics. In the inflationary picture, primordial density perturbations are created from quantum fluctuations, which were stretched on super-horizon scales during inflation and then grew into the structures we see today via gravitational instability.

Perturbations at the surface of last scattering are observable as temperature anisotropies in the CMB. They were detected for the first time by the Cosmic Background Explorer (COBE), a satellite for full sky measurements launched by NASA in 1989. Several experiments followed COBE, balloon-borne experiments as BOOMERanG, MAXIMA and QMAP, ground-based experiments as Saskatoon and ACT, and space experiments as Planck and WMAP. The CMB experiments support the predictions for homogeneity and isotropy of the Universe, and for the quasi-scale-invariance of the primordial fluctuations in the flat Universe [68]. The Gaussianity of the primordial perturbations still needs to be verified. In fact the 7-year WMAP analysis yields the level of non-Gaussianity, which is encoded in the non-linearity parameter $f_{NL}$, in a quite large range, $-10 < f_{NL}^{local} < 74$.[1]

In the simplest inflationary scenario, inflation is driven by a scalar field, called the *inflaton*. This scalar field is responsible for both inflation and the production of density perturbations. The density perturbations are due to the fluctuations of the inflaton when it slowly rolls down its potential and are created about 60 e-folds before the end of inflation. At the end of the inflationary stage, the inflaton field starts to oscillate around the minimum of its potential and decays into other particles, thereby reheating the Universe. Single-field inflation predicts Gaussian density perturbations with an almost scale-invariant spectrum [74, 93]. In fact both the power, $n_s - 1$, of the scalar power spectrum, $\mathcal{P}_s \sim k^{n_s - 1}$, and the non-linearity parameter $f_{NL}$ are proportional to the so-called slow-roll parameters,[2] $\eta$ and $\epsilon$, and these parameters need to be very small, $\epsilon, \eta \ll 1$, in order to have inflation. This prediction is in very good agreement with the present observational data. However the measure of the level of non-Gaussianity is not precise yet and a deviation from Gaussianity is still allowed [68]. Therefore other inflationary models, which differ from the standard paradigm and allow a higher level of non-Gaussianity of the perturbations, may also fit the data.

The precision of the observations improves every year, so in the near future we will be able to distinguish among the myriad different inflationary scenarios. Non-Gaussianity has the potential and the challenge to be a discriminating measure. A simple deviation from the standard scenario is represented by a model of inflation with two scalar fields, the *curvaton scenario* [81, 84]. In this model inflation is driven by the inflaton field, while the curvature perturbations are produced from an initial isocurvature fluctuation associated with the fluctuations of a second scalar field, the *curvaton*. It is assumed that the perturbations given by the inflaton field are negligible. Therefore this scenario liberates the inflaton field from the responsibility of generating the density perturbations and therefore it avoids the conditions on the slow-roll parameters to affect the level of

[1] This result is valid for the "local" shape of the primordial bispectrum. See Section 2.6

[2] $f_{NL} \sim \mathcal{O}(\epsilon, \eta)$ refers to the level of non-Gaussianity which is generated during the inflationary stage and it does not take into account the enhancement of non-Gaussianity due to the gravitational dynamics after inflation. If one takes into account the late-time evolution of the cosmological perturbations, the level of non-Gaussianity is $\sim \mathcal{O}(1)$. See Section 2.8.

non-Gaussianity. Other scenarios where it is possible to produce a large non-Gaussianity are hybrid and multi-brid models [3, 9, 10, 42, 59, 99] and certain modulated and tachyonic (p)re-heating scenarios [36, 40, 60, 129].

Even though inflation successfully solves the main puzzles of the early Universe, we lack a derivation of an inflationary theory from first principles. As proposed by many authors, a good scenario to implement inflation is *supergravity* [66, 128]. In the first part of this thesis we study how to realize the curvaton scenario in supergravity in the context of chaotic inflation. This procedure gives rise to what we called a *supercurvaton scenario*. We study the level of non-Gaussianity generated in this model and show that the non-linearity parameter $f_{NL}$ may take values in the observationally interesting range. Moreover we demonstrate that if inflation is long enough, the average value of the curvaton contribution to the amplitude of metric perturbations and the averaged value of the parameter $f_{NL}$ do not depend on the initial conditions of the curvaton field. Thus, while the curvaton models are more complicated than the single-field inflationary models, they make the resulting scenario much more flexible, which may be important for a proper interpretation of the coming observational data.

Another main riddle of the Universe is the presence of magnetic fields in all celestial objects: planets, stars, galaxies and clusters of galaxies carry fields which are large and extensive [17, 43, 71]. They have an intensity of micro Gauss and are correlated on scales of the order of the galaxy or cluster size. Remarkably, magnetic fields seem to pervade the entire Universe and be present also in the intergalactic medium. Recent data from Fermi and HESS have been used to put a lower bound on the strength of the intergalactic field: $B \gtrsim 10^{-15}$ G [31, 100, 116].

The origin of these fields is unknown. An elaborate magnetohydrodynamical process, called dynamo mechanism, has been proposed to amplify very weak seed fields into the fields observed today in the galaxies. It is based on the conversion of the kinetic energy of an electrically conducting fluid into magnetic field energy. Today, the efficiency of such a mechanism has been brought into question both by improved theoretical work and new observations of magnetic fields in high redshift galaxies. For instance, the fact that high $z$ galaxies have fields comparable to the one of the Milky Way is incompatible with the necessary number of turns in order for the dynamo mechanism to operate. Second, most galaxies and clusters have fields of a few micro Gauss and this is not compatible with the different number of rotations and the parameters involved in every galaxy. In addition, magnetic fields seem to increase with redshift. Furthermore, even if the dynamo mechanism was effective, a seed field is required to initiate the process and the origin of this seed field is still missing. Magnetic fields in clusters of galaxies have strength and coherence size comparable to, and in some cases larger than, galactic fields. In the standard cold dark matter scenario for structure formation, clusters form for aggregation of galaxies. It is now believed that magnetic fields in the intercluster medium cannot form from ejection of the galactic field, thus it seems that we can exclude a common astrophysical origin for both fields [51].

All these facts seem to point in the direction of a cosmological origin of the observed magnetic fields. Cosmological magnetic fields may arise during phase transitions, for example the electroweak [2, 16, 28, 29, 39, 49, 58, 120] or the Quantum-Chromo-Dynamics (QCD) [5, 21, 106] phase transition. Another proposal for the generation of primordial magnetic fields comes from cosmic strings [6, 121, 127]. These mechanisms are proposed either to directly give rise to the magnetic field observed today or to provide the seed field which will be amplified by the dynamo. However, also the seed fields must satisfy two requirements related to their coherence length and amplitude: the coherence size should not drop below 10 Kpc, otherwise it will destabilize the dynamo, and the minimum required strength should vary between $10^{-12}$ and $10^{-22}$ G. The main problem related to these constraints is that magnetic fields generated between inflation and recombination have too small coherence length because of causality, which constrains the field inside the size of the horizon at the time of magnetogenesis.

A prime candidate for the generation of magnetic fields which can solve the scale problem is inflation, since it creates superhorizon-sized correlations [119]. The idea is that inflation can amplify quantum vacuum fluctuations and thus generate long-wavelength magnetic fields. It is known that in the Friedmann-Lemaître-Robertson-Walker (FLRW) Universe the conformal vacuum is preserved if the theory is conformally invariant [102]. Classical electrodynamics is conformally invariant, so that photons should not be produced in cosmological background. Thus the conformal invariance of the electromagnetism must be broken to produce long-wave magnetic fields via excitation of the vacuum fluctuations. Many mechanisms provide the breaking of the conformal invariance of the electromagnetic field [7, 32, 33, 48, 73, 89, 107, 118, 119]. However, if the fields produced during inflation are too strong, they might have a back reaction on the background and thus spoil inflation. In the second part of this thesis the issue of the back reaction of the magnetic field on the background is studied extensively. We show that this back reaction is very important and leads to rather strong bounds on the maximal value of the strength of primordial magnetic fields which seems not enough to explain the observed fields as a result of the amplification of these primordial seeds by dynamo mechanism.

The thesis is organized as follows. In Chapter 2 we review the essential facts about inflation and the linear theory of perturbations. First we discuss the basics about the FLRW cosmology and inflation, focussing on the simplest inflationary model with one scalar field. After reviewing the quantization of a massive free scalar field in unperturbed de Sitter spacetime, we present the basic facts from the theory of cosmological perturbations. Then we very briefly describe the main effect for the generation of CMB anisotropies and explain the idea behind the emergence of non-Gaussianity in the CMB. Finally we show that single-field models of inflation generate negligible non-Gaussianity and we describe how in these models non-Gaussianity is enhanced by the gravitational dynamics after inflation.

In Chapter 3 we introduce inflationary models with two scalar fields. In the first part we give a basic description of the curvaton scenario, explaining the generation of curvature perturbations and calculating the level of non-Gaussianity. In the second part we present the supercurvaton scenario, following our paper [26]. First we describe how to realize chaotic inflation in supergravity. Then we investigate how to implement the curvaton scenario in supergravity in the context of chaotic inflation. Lastly we study the consequences of the supercurvaton scenario, in particular we compute the non-linearity parameter $f_{NL}$ which arises in this class of theories and we find that it can take values in the observationally interesting range from $\mathcal{O}(10)$ to $\mathcal{O}(100)$.

In Chapter 4 we present a brief review of large-scale magnetic fields. We describe the observational methods used to measure them. Then we summarize the observations of their typical strength and coherence size in galaxies, clusters of galaxies and the bounds on the intergalactic magnetic fields. We briefly describe the effects of a magnetic field on the CMB and the constraints on the field amplitude set by Big Bang nucleosynthesis. Finally we discuss several proposals for the generation of the observed magnetic fields in the early Universe.

Chapter 5 deals with the production of magnetic fields during inflation. In the first part we explain why inflation is a good candidate for magnetogenesis and how the generation of large-scale magnetic proceeds. We show that the requirement is the breaking of the conformal invariance of electromagnetism and we review several models proposed in the literature that realize this condition. In the second part, which is based on our paper [27], we consider a very general class of models where the conformal invariance is broken and study the problem of the back reaction of the generated magnetic field on the background. We finally provide the limits on the field strength arising from the requirement that the generated magnetic field does not spoil the dynamics of the background.

# Chapter 2

# Inflation and perturbation theory

One of the most relevant ideas in cosmology is *cosmological inflation*, a period of quasi-exponential expansion in the very early Universe. A model of inflationary type was first proposed by Starobinsky [114] in 1979. This was the first model predicting gravitational waves in the Universe in a quasi-de Sitter stage. In 1981 Mukhanov and Chibisov [97] proposed the first mechanism of production of adiabatic perturbations of the metric, which are responsible for the origin of all cosmological structures that are now visible, like galaxies, clusters of galaxies and the whole web of large-scale structure. According to this mechanism, the large-scale structure originated from tiny density fluctuations, which were stretched to cosmological scales during inflation and then grew into the structures we see today via gravitational instability. In 1981 - 1982 Guth [52], Linde [75], Albrecht and Steinhardt [4] implemented the inflationary scenario in particle physics in order to solve the flatness, the horizon and the monopole problems of the standard Big-Bang cosmology.

In this chapter we review the basic arguments behind the theory of inflation. Sections 2.1 and 2.2 are devoted to the basic equations describing the dynamics of the Friedmann Universe and inflation. In Section 2.3 we describe the idea of the emergence of quantum fluctuations during inflation and in Section 2.4 we present some details about linear perturbation theory. In Section 2.5 we explain how curvature perturbations give rise to the anisotropies in the CMB and in Section 2.6 we sketch the idea behind the non-Gaussianity in the CMB. In Section 2.7 we briefly compute the level of non-Gaussianity in single-field models of inflation and in Section 2.8 we describe how non-Gaussianity is enhanced due to non-linear effects after inflation.

## 2.1 Cosmic evolution

The standard cosmology is based on the assumption that the Universe is homogeneous and isotropic on large scales and it is described by the FLRW metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \equiv g_{\mu\nu}dx^\mu dx^\nu, \qquad (2.1)$$

where $a(t)$ is the scale factor, which characterizes the relative size of spacelike hypersurfaces $\Sigma$ at different times, $k$ is the curvature parameter, $g_{\mu\nu}$ is the metric of spacetime and $x^\mu = (t, r, \theta, \phi)$ are the coordinates of events. $k = 1$ for positively curved $\Sigma$, $k = 0$ for flat $\Sigma$ and $k = -1$ for negatively curved $\Sigma$. The evolution of the Universe depends on the single function $a(t)$, whose form is dictated by the matter content of the Universe through the Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \qquad (2.2)$$

Here $R_{\mu\nu}$ is the Ricci tensor and $R$ is the Ricci scalar

$$R_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \qquad R \equiv g^{\mu\nu}R_{\mu\nu}, \qquad (2.3)$$

where

$$\Gamma^\mu_{\alpha\beta} \equiv \frac{g^{\mu\nu}}{2}[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}] \qquad (2.4)$$

are the Christoffel symbols and commas denote partial derivatives $(\cdots)_{,\mu} \equiv \frac{\partial(\cdots)}{\partial x^\mu}$. $T_{\mu\nu}$ is the energy-momentum tensor and $G$ is the Newton's constant.

On large scales matter can be approximated by a perfect fluid with energy-momentum tensor

$$T^\mu_\nu = g^{\mu\alpha}T_{\alpha\nu} = (\rho + p)u^\mu u_\nu - p\delta^\mu_\nu, \qquad (2.5)$$

where $p$ and $\rho$ are the proper energy density and pressure in the fluid rest frame and $u^\mu$ is the four-velocity of the fluid. The equation of state $p = p(\rho)$ depends on the properties of matter and must be specified. In many cosmological interesting cases $p = \omega\rho$, where $\omega$ is constant.

The evolution of the scale factor $a(t)$ follows from the Einstein field equations. The $(0,0)$ component of the Einstein equation gives

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{2.6}$$

Differentiating this equation with respect to time and using the energy conservation equation $\dot{\rho} + 3H(\rho + p) = 0$ we find

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.7}$$

(2.6) and (2.7) are the Friedmann equations. The Friedmann equations together with the equation of state $p = p(\rho)$ form a complete system of equations that determines the unknown functions $a(t)$ and $\rho(t)$. The solutions, and therefore the future of the Universe, depend on the geometry and also on the equation of state.

## 2.2 Inflation

In order to have an accelerated stage of expansion, the scale factor must satisfy the condition $\ddot{a} > 0$. In this section we will follow [93]. If the strong energy dominance condition, $\rho + 3p > 0$, is satisfied, then from Equation (2.7) we see that $\ddot{a} < 0$ and gravity decelerates the expansion. Therefore for an accelerated expansion the strong energy dominance condition must be violated. This is the case, for example, of a positive cosmological constant, for which $p_\Lambda = -\rho_\Lambda$ and $\rho_\Lambda + 3p_\Lambda = -2\rho_\Lambda$. In this case the solution of the Einstein's equations is a de Sitter Universe and for $t \gg H_\Lambda^{-1}$, the expansion is exponential, $a \propto \exp(H_\Lambda t)$. However a cosmological constant does not describe a successful inflation because it does not possess a smooth graceful exit. In fact, in order to have a graceful exit from inflation, we must allow the Hubble parameter to vary in time.

The simplest model for a successful inflation is implemented by a scalar field $\phi$, called *inflaton*, and described by the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{2.8}$$

where $8\pi G = 1$. This is the sum of the Einstein-Hilbert action and the action of the scalar field with canonical kinetic term. The equation of motion of the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{2.9}$$

This equation is equivalent to the equation for a harmonic oscillator with a friction term proportional to the Hubble parameter $H$. We know that a large friction term decreases the initial velocities and causes a slow-roll regime in which the acceleration can be neglected with respect to the friction term. For a general potential $H \propto \sqrt{V}$, therefore for large values of $V$, we can neglect $\ddot{\phi}$ compared to $3H\dot{\phi}$. Assuming $\dot{\phi}^2 \ll V$, the equation of motion (2.9) becomes

$$3H\dot{\phi} + V_{,\phi} = 0, \tag{2.10}$$

and the Friedmann Equation (2.6) for the scalar field becomes

$$H \simeq \sqrt{\frac{V(\phi)}{3}}, \tag{2.11}$$

where $8\pi G = 1$ and $k = 0$. The assumptions we made, $\dot{\phi}^2 \ll |V|$ and $|\ddot{\phi}| \ll 3H\dot{\phi} \sim |V_{,\phi}|$, can be rephrased into two conditions on the scalar potential:

$$\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1, \qquad \left|\frac{V_{,\phi\phi}}{V}\right| \ll 1. \tag{2.12}$$

These are the so-called *slow-roll conditions*. For a power-law potential $V(\phi) = \lambda\phi^n/n$, both conditions are satisfied for $|\phi| \gg 1$. In this case the resulting scale factor is

$$a(\phi(t)) \simeq a_i \exp\left(\frac{1}{2n}(\phi_i^2 - \phi^2(t))\right). \tag{2.13}$$

The class of inflationary models with the simple potential $V(\phi) = \lambda\phi^n/n$, is

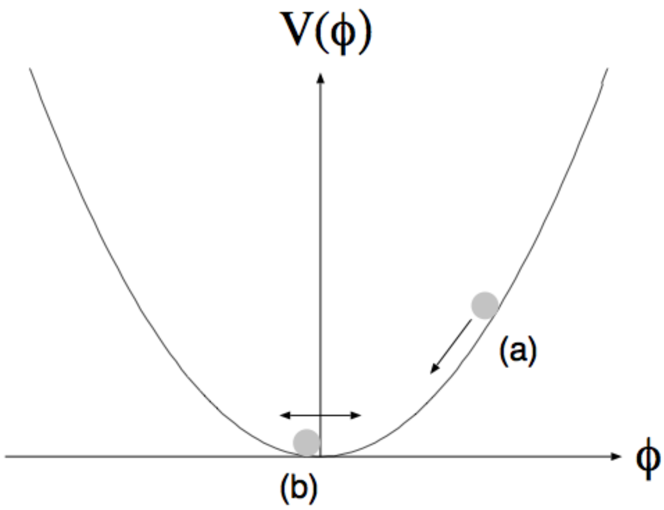


Figure 2.1: Classical evolution of a scalar field $\phi$ with potential $V(\phi)$.

called *chaotic inflation* [76]. After the end of inflation the scalar field begins to oscillate and a deceleration phase starts. A simple model is sketched in figure 2.1, where in the phase (a) the inflaton field $\phi$ rolls down on $V(\phi)$ slowly driving the exponential expansion. In the phase (b) the scalar field oscillates rapidly, ending inflation. During the oscillating phase, the inflaton field decays producing particles and radiation in a process called *reheating*.

Through the years many inflationary models have been proposed. Single-field models differ by the type of potential or by the underlying particle physics theory. There are mainly three broad classes of scenarios: "small field", "large field" and "hybrid" models. An alternative to the single-field inflationary models is represented by models of inflation with several scalar fields. Among the multi-field models the simplest proposal is the curvaton scenario, a model of inflation with two scalar fields, which we will discuss in the next chapter.

## 2.3 Quantum fluctuations in de Sitter spacetime

The inflationary scenario predicts the emergence of quantum vacuum fluctuations in the early Universe. Associated to these quantum fluctuations there are primordial energy density perturbations which are the origin of the large-scale structure. Our current understanding is that during the matter era primordial density inhomogeneities were amplified by gravitational instability and grew into the structures we observe today.

The basic idea behind quantum fluctuations generation is described by the second quantization of a massive free scalar field in unperturbed de Sitter spacetime. We describe the quantization procedure following [18, 69, 94].

In a de Sitter Universe the scale factor evolves as $a(t) = a_0 e^{Ht}$, where $H$ is the time-independent Hubble parameter. Defining the *conformal time*

$$\eta = -\int_t^{\infty} \frac{dt'}{a(t')} = -\exp(-Ht), \tag{2.14}$$

we can rewrite the metric (2.1) ($k = 0$) as

$$ds^2 = \frac{1}{H^2\eta^2}[d\eta^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{2.15}$$

where $-\infty < \eta < 0$ and $0 \leqslant r < \infty$.

Let us consider the case of a massive scalar field $\phi(\boldsymbol{x}, \eta)$. We can write the field using the creation, $\hat{a}^\dagger_{\boldsymbol{k}}$, and annihilation, $\hat{a}_{\boldsymbol{k}}$, operators, with commutation relation $[\hat{a}^\dagger_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}}] = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')$:

$$\phi(\boldsymbol{x}, \eta) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \left[\hat{a}_{\boldsymbol{k}}\varphi_k(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{k}}\varphi^*_k(\eta)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]. \tag{2.16}$$

The canonical commutation relation between $\phi$ and the conjugate momentum $\pi_\phi = a^2(\eta)\phi'$:

$$[\phi(\boldsymbol{x}_1, \eta), \pi_\phi(\boldsymbol{x}_2, \eta)] = a^2(\eta)\,[\phi(\boldsymbol{x}_1, \eta), \phi'(\boldsymbol{x}_2, \eta)] = i\delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{x}_2), \tag{2.17}$$

gives a normalization condition on $\varphi_k(\eta)$, $a^2(\varphi_k{\varphi'}^*_k - \varphi^*_k{\varphi'}_k) = i$. The prime $'$ denotes the derivative with respect to the conformal time, $' = \partial/\partial\eta$. The normalization condition motivates the introduction of an auxiliary field $\chi_k = a\varphi_k$, which satisfies a new normalization condition $\chi_k {\chi'}^*_k - \chi^*_k {\chi'}_k = i$. The new mode functions satisfy the Klein-Gordon equation

$$\chi''_k(\eta) + [k^2 + m^2_\chi(\eta)]\chi_k(\eta) = 0, \tag{2.18}$$

where $m^2_\chi(\eta)$ is a time-dependent effective mass

$$m^2_\chi(\eta) \equiv (m^2 - 2H^2)a^2(\eta) = m^2 a^2(\eta) - \frac{2}{\eta^2}. \tag{2.19}$$

The solution for the Equation (2.18) is

$$\chi_k(\eta) = \sqrt{-\eta}\left[C_1 H^{(1)}_\nu(-k\eta) + C_2 H^{(2)}_\nu(-k\eta)\right], \tag{2.20}$$

where $C_1$ and $C_2$ are integration constants and $\nu^2 = 9/4 - m^2/H^2$. $H^{(1)}_\nu(x)$ is a Hankel function of the first kind and $H^{(2)}_\nu(x) = \left[H^{(1)}_\nu(x)\right]^*$. The particular linear combination used to define the modes determines the choice of the vacuum. In order to fix the integration constants $C_1$ and $C_2$, we define a vacuum state in the remote past $\eta \to -\infty$. In the remote past, when $k|\eta| \gg 1$, the modes do not feel the curvature and we can fix the initial conditions by requiring

$$\chi_k(\eta \to -\infty) \longrightarrow \sqrt{\frac{2}{\pi k}}\left(C_1 e^{-ik\eta} + C_2 e^{ik\eta}\right). \tag{2.21}$$

The second term has negative frequency, so that $C_2 = 0$. The first term with $C_1 = \sqrt{\pi}/2$ gives $\chi_k(\eta) = (2k)^{-1/2} e^{-ik\eta}$. Therefore we find that the functions

$$\chi_k(\eta) = \frac{\sqrt{-\pi\eta}}{2} H^{(1)}_\nu(-k\eta) \tag{2.22}$$

have the required asymptotic behaviour. Thus the solution for $\varphi_k(\eta)$ is

$$\varphi_k(\eta) = \frac{\sqrt{-\pi\eta}}{2a(\eta)} H^{(1)}_\nu(-k\eta), \tag{2.23}$$

and the one for $\phi(\boldsymbol{x},\eta)$ becomes

$$\phi(\boldsymbol{x},\eta) = \frac{\sqrt{-\pi\eta}}{2a(\eta)} \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \left[\hat{a}_{\boldsymbol{k}} H_\nu^{(1)}(-k\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}_{\boldsymbol{k}}^\dagger H_\nu^{(2)}(-k\eta) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]. \tag{2.24}$$

The annihilation operator $\hat{a}_{\boldsymbol{k}}$ annihilates the vacuum, $\hat{a}_{\boldsymbol{k}}|0_{in}\rangle = 0$. We calculate the amplitude of ground-state fluctuations in this vacuum:

$$\begin{aligned} \langle 0_{in}|\phi^\dagger(\boldsymbol{x},\eta)\phi(\boldsymbol{x},\eta)|0_{in}\rangle &= \int_0^\infty \frac{k^2 dk}{2\pi^2} |\varphi_k(\eta)|^2 \\ &= \frac{-\eta}{8\pi a^2(\eta)} \int_0^\infty k^2 dk \left|H_\nu^{(1)}(-k\eta)\right|^2. \end{aligned} \tag{2.25}$$

Therefore the spectrum of quantum fluctuations at a given comoving wavelength $2\pi k^{-1}$ is

$$\delta_\phi^2(k) \equiv \frac{k^3}{2\pi^2} |\varphi_k(\eta)|^2 = \frac{H^2}{8\pi} (-k\eta)^3 \left|H_\nu^{(1)}(-k\eta)\right|^2. \tag{2.26}$$

These results are valid on all scales. However during inflation $-k\eta = k/(aH)$ becomes very small and the mode leaves the Hubble-horizon $H^{-1}$. Therefore we focus on the behaviour of the fluctuation spectrum of $\phi$ on super-horizon scales. We obtain the spectrum of the $\phi$ fluctuations on super-horizon scales from Equation (2.26) using the asymptotic form of the Hankel function

$$H_\nu^{(1)}(x \ll 1) \approx -i\frac{\Gamma(\nu)}{\pi} \left(\frac{x}{2}\right)^{-\nu}. \tag{2.27}$$

The spectrum is

$$\delta_\phi^2(k) \approx \left(\frac{H}{2\pi}\right)^2 2^{2\nu-3} \left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2 \left(\frac{k}{aH}\right)^{3-2\nu}, \tag{2.28}$$

where $\nu^2 = 9/4 - m^2/H^2$. In the special case $\nu = 3/2$, the spectrum is scale invariant, independent of $k$, $\delta_\phi^2(k) = H^2/(2\pi)^2$. If we assume that $m^2/H^2 \ll 1$, then $\nu = 3/2 - m^2/(3H^2) + \mathcal{O}(m^4/H^4)$ and we obtain

$$\delta_\phi^2(k) \simeq \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{2m^2/(3H^2)}, \tag{2.29}$$

and the spectral index

$$\frac{d\ln\delta_\phi^2}{d\ln k} \simeq \frac{2m^2}{3H^2}. \tag{2.30}$$

In the case of a finite mass, the spectrum would be slightly "blue". However, since $m^2 \ll H^2$, the spectrum is almost scale-invariant. The assumption $m^2 \ll H^2$ determines a long-lasting inflation and makes the scalar field $\phi$ not to roll down the potential too quickly.

In realistic models of inflation the Hubble parameter $H$ changes slightly in time and decreases towards the end of inflation. Let us now investigate how this influences the amplitude of the large-scale fluctuations. The massive scalar field acquires a negative effective mass-squared $\Delta m^2 < 0$ as it is shown in [98]:

$$m_\chi^2(\eta) = -\frac{H}{\phi'}\frac{d^2(\phi'/H)}{d\eta^2} \simeq \left(\frac{d^2V}{d\phi^2} + 9\frac{dH}{dt}\right)a^2(\eta) - \frac{2}{\eta^2} = m_{\text{eff}}^2 a^2(\eta) - \frac{2}{\eta^2}. \tag{2.31}$$

Using the first-order slow-roll approximation, in the case of a quadratic potential $V(\phi) = m^2\phi^2/2$, we have $dH/dt \simeq -(dV/d\phi)^2/(6V) = -m^2/3$, thus $m_{\text{eff}}^2 = m^2 + \Delta m^2 \simeq -2m^2 < 0$ and the spectrum is "red" with spectral index

$$\frac{d\ln\delta_\phi^2}{d\ln k} = \frac{2m_{\text{eff}}^2}{3H^2} \simeq -\frac{4m^2}{3H^2} < 0. \tag{2.32}$$

Any power-law potential $V(\phi) = A\phi^n$ with $A$ positive gives a negative $m_{\text{eff}}^2$ through $m_{\text{eff}}^2 = -(1 + n/2)nA\phi^{n-2}$. For a generic scalar field with arbitrary potential we have

$$m_{\text{eff}}^2 = \frac{d^2V}{d\phi^2} + 9\frac{dH}{dt} = 6\frac{dH}{dt} - 3H\frac{d^2\phi/dt}{d\phi/dt}, \tag{2.33}$$

and the spectral index is

$$\frac{d\ln\delta_\phi^2}{d\ln k} = 4\frac{dH/dt}{H^2} - 2\frac{d^2\phi/dt^2}{H(d\phi/dt)}. \tag{2.34}$$

## 2.4 Linear perturbation theory

In the previous section we have seen the generation of scalar field fluctuations in an unperturbed de Sitter spacetime (no perturbations in the metric). However scalar field fluctuations perturb the energy-momentum tensor producing metric perturbations. In this section we study linear perturbation theory in inflation, which includes perturbations of the metric and of a scalar field. We follow the treatment of [93].

### 2.4.1 Classification of the perturbations

The metric of a flat FLRW Universe with small perturbations can be written as

$$ds^2 = (\bar{g}_{\alpha\beta} + \delta g_{\alpha\beta})dx^\alpha dx^\beta, \tag{2.35}$$

where $|\delta g_{\alpha\beta}| \ll \bar{g}_{\alpha\beta}$. The spatially flat, homogeneous and isotropic background spacetime possesses a number of symmetries and these symmetries allow the metric perturbations $\delta g_{\alpha\beta}$ to be categorized into three different types: scalar, vector and tensor perturbations. At a given moment of time the background is invariant with respect to the group of spatial rotations and translations. The $\delta g_{00}$ component behaves as a scalar under these rotations and hence

$$\delta g_{00} = 2a^2\Phi_P, \tag{2.36}$$

where $\Phi_P$ is a 3-scalar. The spacetime components $\delta g_{0i}$ can be decomposed into the sum of the spatial gradient of some scalar $B$ and a vector $S_i$ with zero divergence

$$\delta g_{0i} = a^2(B_{,i} + S_i), \tag{2.37}$$

where $B_{,i} = \partial B/\partial x^i$. The components $\delta g_{ij}$, which behave as a tensor under 3-rotations, can be written as the sum of irreducible pieces:

$$\delta g_{ij} = a^2(2\Psi_P\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}), \tag{2.38}$$

where $\Psi_P$ and $E$ are scalar functions, $F^i_{,i} = 0$ and $h_{ij}$ is traceless and transverse, namely $h^i_i = 0$, $h^i_{j,i} = 0$.

Scalar perturbations are characterized by the four scalar functions $\Phi_P$, $\Psi_P$, $B$ and $E$. They are caused by energy density inhomogeneities. They exhibit gravitational instability and may lead to the formation of structure in the Universe. Vector perturbations are described by the two vectors $S_i$ and $F_i$ and are related to the rotational motion of the fluid. As in Newtonian theory, they decay very quickly and are not very interesting from the point of view of cosmology. Tensor perturbations $h_{ij}$ describe gravitational waves, which are the degrees of freedom of the gravitational field itself. In the linear approximation the gravitational waves do not induce any perturbations in the perfect fluid.

Using conformal time, the metric for scalar perturbations takes the form

$$ds^2 = a^2\left[(1+2\Phi_P)d\eta^2 + 2B_{,i}dx^i d\eta - \left((1-2\Psi_P)\delta_{ij} - 2E_{,ij}\right)dx^i dx^j\right]. \tag{2.39}$$

For vector perturbations

$$ds^2 = a^2\left[d\eta^2 + 2S_i dx^i d\eta - \left(\delta_{ij} - F_{i,j} - F_{j,i}\right)dx^i dx^j\right], \tag{2.40}$$

and for tensor perturbations

$$ds^2 = a^2\left[d\eta^2 - \left(\delta_{ij} - h_{ij}\right)dx^i dx^j\right]. \tag{2.41}$$

Scalar perturbations change under a change of coordinates. Under the gauge transformation

$$x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha, \tag{2.42}$$

and decomposing the spatial component as $\xi^i = \xi^i_\perp + \zeta^{,i}$, the scalar metric perturbations transform as

$$
\begin{aligned}
\Phi_P \rightarrow \tilde{\Phi}_P &= \Phi_P - \frac{1}{a}(a\xi^0)', \\
\Psi_P \rightarrow \tilde{\Psi}_P &= \Psi_P + \frac{a'}{a}\xi^0, \\
B \rightarrow \tilde{B} &= B + \zeta' - \xi^0, \\
E \rightarrow \tilde{E} &= E + \xi.
\end{aligned}
\tag{2.43}
$$

By choosing $\xi^0$ and $\zeta$ appropriately we can make any two of the four functions $\Phi_P$, $\Psi_P$, $B$ and $E$ vanish. The simplest gauge-invariant linear combinations of these functions are

$$\Phi \equiv \Phi_P - \frac{1}{a}\left[a(B - E')\right]', \qquad \Psi \equiv \Psi_P + \frac{a'}{a}(B - E'). \tag{2.44}$$

### 2.4.2 Evolution of the perturbations

In order to obtain the equations for the perturbations we need to linearize the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R\delta_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.45}$$

for small inhomogeneities about a FLRW Universe. The Einstein tensor for the background metric is

$$\overline{G}^0_0 = \frac{3\mathcal{H}}{a^2}, \qquad \overline{G}^0_i = 0, \qquad \overline{G}^i_j = \frac{1}{a^2}(2\mathcal{H}' + \mathcal{H}^2)\delta^i_j, \tag{2.46}$$

where $\mathcal{H} \equiv a'/a$. In order to satisfy the background Einstein equations, the energy-momentum tensor for matter, $\overline{T}^\alpha_\beta$ , must have the following symmetry properties:

$$\overline{T}^0_i = 0, \quad \overline{T}^i_j \propto \delta^i_j. \tag{2.47}$$

In the presence of small perturbations the Einstein tensor can be written as $G^\alpha_\beta = \overline{G}^\alpha_\beta + \delta G^\alpha_\beta + \cdots$, where $\delta G^\alpha_\beta$ are terms which are linear in the metric perturbations. The energy-momentum tensor can be split in the same way and the linearized Einstein equations are

$$\overline{\delta G}^{\alpha}_{\beta} = 8\pi G \overline{\delta T}^{\alpha}_{\beta}. \tag{2.48}$$

### 2.4.3 Perturbations of a slowly-rolling scalar field

Let us consider the Universe filled by a scalar field $\phi$ with potential $V(\phi)$. A small perturbation in the scalar field $\phi = \phi_0(\eta) + \delta\phi(\boldsymbol{x}, \eta)$ induces metric perturbations and the metric takes the form (2.39). The homogeneous component satisfies the Klein-Gordon equation

$$\phi_0'' + 2\mathcal{H}\phi_0' + a^2 V_{,\phi} = 0. \tag{2.49}$$

To linear order in the metric and the field perturbations, it becomes

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \Delta(\delta\phi - \delta\phi_0'(B - E')) + a^2 V_{,\phi\phi}\delta\phi - \phi_0'(3\Psi + \Phi)' + 2a^2 V_{,\phi}\Phi = 0. \tag{2.50}$$

This equation is valid in any coordinate system. We can rewrite it in terms of the gauge-invariant variables $\Phi$ and $\Psi$ from Equation (2.44) and the gauge-invariant scalar field perturbation

$$\overline{\delta\phi} \equiv \delta\phi - \phi_0'(B - E'). \tag{2.51}$$

The resulting equation is

$$\overline{\delta\phi}'' + 2\mathcal{H}\overline{\delta\phi}' - \Delta\overline{\delta\phi} + a^2 V_{,\phi\phi}\overline{\delta\phi} - \phi_0(3\Psi + \Phi)' + 2a^2 V_{,\phi}\Phi = 0. \tag{2.52}$$

In order to find the three unknown variables $\overline{\delta\phi}$, $\Phi$ and $\Psi$, this equation must be supplemented by the Einstein equations. The energy-momentum tensor for the scalar field is

$$T^{\alpha}_{\beta} = g^{\alpha\gamma}\phi_{,\gamma}\phi_{,\beta} - (g^{\gamma\delta}\phi_{,\gamma}\phi_{,\delta} - V(\phi))\delta^{\alpha}_{\beta}, \tag{2.53}$$

and hence its $(0, i)$ component is

$$\overline{\delta T}^{0}_{i} = \frac{1}{a^2}\phi'_0\delta\phi_{,i} - \frac{1}{a^2}\phi'^2_0(B - E')_{,i} = \frac{1}{a^2}(\phi'_0\overline{\delta\phi})_{,i}. \tag{2.54}$$

The $(0, i)$ component of the Einstein equations $\overline{\delta G}^{\alpha}_{\beta} = 8\pi G \overline{\delta T}^{\alpha}_{\beta}$ is

$$(\Psi' + \mathcal{H}\Psi)_{,i} = 4\pi G a^2 \overline{\delta T}^{0}_{i}. \tag{2.55}$$

Therefore for a scalar field, this equation becomes

$$\Psi' + \mathcal{H}\Phi = 4\pi G \phi_0' \overline{\delta\phi}. \tag{2.56}$$

The non-diagonal spatial component of the Einstein equations

$$\begin{aligned}&\left[\Psi'' + \mathcal{H}(2\Psi + \Phi)' + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{2}\Delta(\Phi - \Psi)\right]\delta^i_j \\ &- \frac{1}{2}(\Phi - \Psi)_{,ij} = -4\pi G a^2 \overline{\delta T}^{i}_{j},\end{aligned} \tag{2.57}$$

where $\overline{\delta T}^{i}_{j} = -\overline{\delta p}\delta^i_j$, reduces to

$$(\Phi - \Psi)_{,ij} = 0 \qquad (i \neq j). \tag{2.58}$$

The only solution consistent with $\Psi$ and $\Phi$ being perturbations is $\Psi = \Phi$.

Now we determine the behaviour of the long-wavelength perturbations using the slow-roll approximation. To do so, first we rewrite Equations (2.52) and (2.56) in terms of physical time:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \Delta\delta\phi + V_{,\phi\phi}\delta\phi - 4\dot{\phi}_0\dot{\Phi} + 2V_{,\phi}\Phi = 0, \tag{2.59}$$

$$\dot{\Phi} + H\Phi = 4\pi\dot{\phi}_0\delta\phi, \tag{2.60}$$

where we have set $\overline{\delta\phi} \equiv \delta\phi$ and we have used $\Phi = \Psi$, and where $G = 1$. The spatial derivative term $\Delta\delta\phi$ can be neglected for long-wavelength perturbations. Since we are considering the slow-roll approximation, we next omit terms proportional to $\delta\ddot{\phi}$ and $\dot{\Phi}$ and after finding the solution of the simplified equations one can check that the omitted terms are actually negligible. The new equations become

$$3H\delta\dot{\phi} + V_{,\phi\phi}\delta\phi + 2V_{,\phi}\Phi \simeq 0, \tag{2.61}$$

$$H\Phi \simeq 4\pi\dot{\phi}_0\delta\phi. \tag{2.62}$$

Introducing the new variable $y \equiv \delta\phi/V_{,\phi}$, we obtain

$$3H\dot{y} + 2\Phi \simeq 0, \tag{2.63}$$

$$H\Phi \simeq 4\pi\dot{V}y. \tag{2.64}$$

During inflation $3H^2 \simeq 8\pi V$, therefore

$$\frac{d(yV)}{dt} = 0, \tag{2.65}$$

which gives $y = A/V$, where $A$ is a constant of integration. The final result is

$$\delta\phi_k = A_k\frac{V_{,\phi}}{V}, \tag{2.66}$$

$$\Phi_k = 4\pi A_k\frac{\dot{\phi}_0}{H}\frac{V_{,\phi}}{V} = -\frac{1}{2}A_k\left(\frac{V_{,\phi}}{V}\right)^2. \tag{2.67}$$

The integration constant is fixed by requiring that at the moment of horizon crossing, $\delta\phi_k$ has the minimal vacuum amplitude and one finds that

$$A_k \sim \frac{k^{-1/2}}{a_k}\left(\frac{V}{V_{,\phi}}\right)_{k\sim Ha}. \tag{2.68}$$

At the end of inflation ($t \sim t_f$), the slow-roll conditions are not valid anymore and $V_{,\phi}/V$ becomes of order unity. Therefore the amplitude of the metric fluctuations becomes

$$\delta_\Phi(k, t_f) \sim A_k k^{3/2} \sim \left(H\frac{V}{V_{,\phi}}\right)_{k\sim aH} \sim \left(\frac{V^{3/2}}{V_{,\phi}}\right)_{k\sim aH}. \tag{2.69}$$

Using $H \simeq \sqrt{V/3}$ and $V_{,\phi} = -3H\dot{\phi}$ one can show that this result is consistent with the spectrum $\Delta_\zeta^2$ in [69], Equation (2.43)

$$\delta_\Phi^2 \sim \Delta_\zeta^2 = \left[\frac{H^2}{2\pi\phi'}\right]. \tag{2.70}$$

The relation between the variables $\Phi$ and $\zeta$ will be explained in the following.

In the case of a power-law potential $V = \lambda\phi^n/n$ we have

$$\delta_\Phi(k, t_f) \sim \lambda^{1/2}(\phi^2_{k\sim aH})^{(n+2)/4} \sim \lambda^{1/2}(\ln\lambda_{ph}H_k)^{(n+2)/4}, \tag{2.71}$$

where $\lambda_{ph} \sim a(t_f)k^{-1}$ is the physical wavelength. In the specific case $V = m^2\phi^2/2$ the amplitude of the metric fluctuations is

$$\delta_\Phi \sim m\ln(\lambda_{ph}H_k). \tag{2.72}$$

Perturbations present at the end of inflation do not change during reheating. Therefore the shape of the spectrum is predicted: it has logarithmic deviations from a flat spectrum with the amplitude growing slightly towards larger scales.

The prediction of a slightly red-tilted spectrum is in agreement with the current observational data. In fact CMB experiments [68] show that a red-tilted primordial power spectrum is preferred. The 7-year WMAP data combined with the latest distance measurements from the Baryon Acoustic Oscillations (BAO) in the distribution of galaxies [110] and the Hubble constant ($H_0$) measurement [111] exclude a scale-invariant spectrum at 99.5% confidence level, if they ignore tensor modes (gravitational waves). The spectral index $n_s$ defined as

$$n_s - 1 \equiv \frac{d\ln\delta_\phi^2}{d\ln k} \tag{2.73}$$

is found to be $n_s = 0.968 \pm 0.012$ at 68% confidence level.

### 2.4.4 Gauge-invariant variables

We have seen that the value of $\overline{\delta\phi}$, (2.51), and $\Psi$, (2.44) are independent of a gauge choice. Using a proper combination of these gauge-invariant variables, one can build a new gauge-invariant variable

$$v \equiv a\left(\overline{\delta\phi} + \frac{\phi_0'}{\mathcal{H}}\Psi\right). \tag{2.74}$$

Let us note that, as we have seen before, in the case of a Universe filled with a single scalar field the non-diagonal spatial component of the Einstein Equations (2.57) gives $\Psi = \Phi$. The new gauge-invariant variable $v$ plays an important role. In fact Mukhanov, Feldman and Brandenberger [98] showed that $v$ obeys the same Klein-Gordon equation as the variable $\chi_k$ that we defined in Section 2.3 about quantization of quantum vacuum fluctuations during inflation, see Equation (2.18). Therefore the argument on the quantization of quantum fluctuations during inflation applies to $v$ as well.

Another gauge-invariant variable was proposed by Bardeen, Steinhardt and Turner [8]

$$\zeta \equiv -\frac{Hv}{\phi_0'} = -\Psi - \frac{H}{\dot{\phi}_0}\overline{\delta\phi}. \tag{2.75}$$

The variable $\zeta$ is important in the perturbation analysis because it is gauge-invariant and also because it is conserved on super-horizon scales throughout the cosmic evolution. Here again $\Psi = \Phi$ in the case of a single scalar field.

The variable $\Phi$ gives the closest analogy to the Newtonian potential. The general relativistic cosmological perturbation theory gives the relation between $\zeta$ and $\Phi$ for adiabatic perturbations [93],

$$\zeta = \frac{2}{3}\frac{\mathcal{H}^{-1}\Phi' + \Phi}{1+\omega} + \Phi, \tag{2.76}$$

where $\omega$ is the equation of state parameter. On super-horizon scales during the radiation era ($\omega = 1/3$) we have $\Phi = \frac{2}{3}\zeta$ and during the matter era ($\omega = 0$) we have $\Phi = \frac{3}{5}\zeta$.

## 2.5 Generation of CMB anisotropy

Temperature fluctuations in the CMB arise due to five distinct physical effects: our peculiar velocity with respect to the cosmic rest frame, fluctuations in the gravitational potential on the surface of last scattering, fluctuations intrinsic to the radiation field itself on the surface of last scattering, the peculiar velocity of the surface of last scattering and the damping of anisotropies if the Universe should be re-ionized after decoupling. The second effect, known as *Sachs-Wolfe* effect, is the dominant contribution to the anisotropies on large angular scales, $\theta \gg 1^\circ$. Let us therefore consider this effect.

The Sachs-Wolfe effect predicts that the CMB that resides in a potential well initially has an adiabatic temperature fluctuation of $\Delta T/T = [2/3(1+\omega)]\Phi$, and at the decoupling epoch, when it is climbing up the potential, it receives an additional fluctuation $-\Phi$. The total CMB fluctuation that we observe today is therefore

$$\frac{\Delta T}{T} = \frac{2}{3(1+\omega)}\Phi - \Phi = -\frac{1+3\omega}{3+3\omega}\Phi = -\frac{1+3\omega}{5+3\omega}\zeta. \tag{2.77}$$

In the case of isocurvature perturbations the initial temperature fluctuations are given by $-\Phi$ for both radiation and matter era, therefore the total amount of CMB perturbations is given by $\Delta T/T = -\Phi - \Phi = -2\Phi$.

At the decoupling epoch, the Universe has already entered the matter era with $\omega = 0$ and $\Delta T/T = -\frac{1}{3}\Phi = -\frac{1}{5}\zeta$ and, using (2.70), the spectrum of the Sachs-Wolfe effect is

$$\delta^2_{SW}(k) = \frac{1}{25}\delta^2_\zeta(k) = \left[\frac{H^2}{10\pi\dot{\phi}}\right]^2, \tag{2.78}$$

where $H$ is the Hubble parameter during inflation [19]. We can obtain the angular power spectrum $C_l$ by projecting the 3-dimensional CMB fluctuation spectrum $\delta^2_{SW}(k)$ on the sky,

$$\begin{aligned} C_l^{SW} &= 4\pi \int_0^\infty \frac{dk}{k}\delta^2_{SW}(k) j_l^2[k(\eta_0 - \eta_{dec})] \\ &= C_2^{SW}\frac{\Gamma[(9-n)/2]\Gamma[l+(n-1)/2]}{\Gamma[(n+3)/2]\Gamma[l+(5-n)/2]}, \end{aligned} \tag{2.79}$$

where $\eta_0$ and $\eta_{dec}$ are the present day and the decoupling time, and $n \equiv 1 + [d \ln \delta^2(k)/d \ln k] \equiv n_s$ is the spectral index defined in (2.73).

## 2.6 Non-Gaussian fluctuations

While quantum fluctuations are Gaussian, non-linearity in inflation produces weakly non-Gaussian fluctuations, which result in non-Gaussianity in the CMB. In the following we present the idea behind it reviewing [69].

The curvature perturbation produces small anisotropies $\Delta T/T$ in the CMB. In linear perturbation theory the relation between $\Phi$ and $\Delta T/T$ is linear,

$$\frac{\Delta T}{T} \sim g_T \Phi, \tag{2.80}$$

where $g_T$ is the radiation transfer function. We have seen in the previous section that for temperature fluctuations on super-horizon scales at the decoupling epoch, $g_T = -1/3$ for adiabatic fluctuations and $g_T = -2$ for isocurvature fluctuations.

According to the general relativistic perturbation theory there is a non-linear relation between $\Delta T/T$ and $\Phi$:

$$\frac{\Delta T}{T} \sim (g_T \Phi + f_\Phi \Phi^2), \tag{2.81}$$

where $f_\Phi \sim \mathcal{O}(1)$ is the higher second order correction arising from the second-order perturbation theory [105]. So even if $\Phi$ is Gaussian, $\Delta T/T$ is weakly non-Gaussian.

We should also note that non-linearity in inflation makes $\Phi$ non-Gaussian. By expanding the fluctuation dynamics in inflation up to second order, we find a non-linear relation between $\Phi$ and the inflaton fluctuations $\delta\phi$,

$$\Phi \sim M_P^{-1} g_\Phi (\delta\phi + M_P^{-1} f_{\delta\phi} \delta\phi^2). \tag{2.82}$$

Salopek and Bond [112] showed that this relation is a solution for curvature perturbations on super-horizon scales. The solution gives $g_\Phi \sim \mathcal{O}(10)$ and $f_{\delta\phi} \sim \mathcal{O}(10^{-1})$ for a class of slowly-rolling single-field inflationary models.

Quantum fluctuations produce Gaussian $\delta\phi$. However, non-trivial interaction terms in the equation of motion of the inflaton field, or a non-linear coupling between long-wavelength classical fluctuations and short-wavelength quantum fluctuations (in the context of chaotic inflation), can make $\delta\phi$ weakly non-Gaussian:

$$\delta\phi \sim g_{\delta\phi}(\epsilon + M_P^{-1} f_\epsilon \epsilon^2), \tag{2.83}$$

where $\epsilon$ are initially produced quantum fluctuations, $g_{\delta\phi} \sim 1$ and $f_\epsilon \sim \mathcal{O}(10^{-1})$.

The above contributions result in a non-linear relation between $\Delta T/T$ and $\Phi$,

$$\frac{\Delta T}{T} \sim g_T \left[\Phi_L + (f_\Phi + g_\Phi^{-1} f_{\delta\phi} + g_\Phi^{-1} g_{\delta\phi}^{-1} f_\epsilon)\Phi_L^2\right], \tag{2.84}$$

where $\Phi_L \equiv g_\Phi g_{\delta\phi} M_P^{-1}\epsilon \sim 10 M_P^{-1}\epsilon$ is an auxiliary Gaussian curvature perturbation. We can define a non-linear coupling parameter $f_{NL} = f_\Phi + g_\Phi^{-1} f_{\delta\phi} + g_\Phi^{-1} g_{\delta\phi}^{-1} f_\epsilon$, where the first term is $\mathcal{O}(1)$ and it is dominant compared to the other two terms, which are $\mathcal{O}(10^{-2})$, non-linearity in slow-roll. Using $f_{NL}$ we can rewrite (2.84) and (2.80) as

$$\Phi(\boldsymbol{x}) = \Phi_L(\boldsymbol{x}) + f_{NL}\left[\Phi_L^2(\boldsymbol{x}) - \langle\Phi_L^2(\boldsymbol{x})\rangle\right], \tag{2.85}$$

where the angular brackets denote the statistical ensemble average. $f_{NL}$ is a dimensionless parameter and, as it is defined in (2.85), it is used to characterize the *local* form of Gaussianity.

We can define a generalized $f_{NL}$ parameter in the following way:

$$f_{NL}(k_1, k_2, k_3) = \frac{B_\Phi(k_1, k_2, k_3)}{2[P_\Phi(k_1)P_\Phi(k_2) + P_\Phi(k_2)P_\Phi(k_3) + P_\Phi(k_3)P_\Phi(k_1)]}, \tag{2.86}$$

where $P_\Phi(k)$ and $B_\Phi(k_1, k_2, k_3)$ are respectively the power spectrum and the bispectrum and they are defined in the Fourier space as

$$\langle\Phi_{\boldsymbol{k}_1}\Phi_{\boldsymbol{k}_2}\rangle = (2\pi)^3 P_\Phi(k_1)\delta^3(\boldsymbol{k}_1 + \boldsymbol{k}_2), \tag{2.87}$$

$$\langle\Phi_{\boldsymbol{k}_1}\Phi_{\boldsymbol{k}_2}\Phi_{\boldsymbol{k}_3}\rangle = (2\pi)^3 B_\Phi(k_1, k_2, k_3)\delta^3(\boldsymbol{k}_1 + \boldsymbol{k}_2). \tag{2.88}$$

The delta function in Equation (2.88) enforces that the three Fourier modes of the bispectrum form a closed triangle. Different inflationary models predict maximal non-Gaussianity for different triangle configurations. Physically motivated models for producing non-Gaussian perturbations often produce signals that peak at special triangle configurations. Three important special cases are: the *squeezed triangle* ($k_1 \approx k_2 \gg k_3$), this is the dominant mode of models with multiple light fields during inflation, the curvaton scenario, inhomogeneous reheating and New Ekpyrotic models. The *equilateral triangle* ($k_1 = k_2 = k_3$), this is relevant for models with higher-derivative interactions and non-trivial speeds of sound. The *folded triangle* ($k_1 = 2k_2 = 2k_3$), this shape arises in models with non-standard initial states.

## 2.7 Non-Gaussianity in single-field inflation

Creminelli and Zaldarriaga [23], generalizing an observation by Maldacena [86] imposed a consistency relation on the 3-point correlation function of single-field inflation. They demonstrated that the non-linearity parameter $f_{NL}$ is proportional to $1 - n_S$, where $n_S$ is the spectral index of scalar perturbations, and thus $f_{NL}$ for single-field models is very small. Now we review the argument of Creminelli and Zaldarriaga [23] as it is treated in [45].

We want to calculate the 3-point correlation function $\langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \rangle$[1] in the limit $k_3 \ll k_1, k_2$. We have that

$$\langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \rangle = \langle \langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \rangle_{\zeta_{\boldsymbol{k}_3}} \zeta_{\boldsymbol{k}_3} \rangle, \tag{2.89}$$

where we define $\langle \cdots \rangle_{\zeta_{\boldsymbol{k}_3}}$ to be the expectation value of $\cdots$ given that $\zeta_{\boldsymbol{k}_3}$ has a particular value. We compute $\langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \rangle_{\zeta_{\boldsymbol{k}_3}}$ after the $k_1, k_2$ modes have left the horizon so that the mode $k_3$ will have crossed the horizon in the distant past. Thus $\zeta_{\boldsymbol{k}_3}$ will be part of an essentially classical background $\zeta^B$ which affects the scalar field through the metric. Considering only modes far outside the horizon, the metric is

$$ds^2 = dt^2 - a^2(t) e^{2\zeta^B(\boldsymbol{x})} dx^2, \tag{2.90}$$

[1]In this case the variable $\zeta$ is a gauge dependent variable and it is different from the one used in 2.4.4. At linear level it corresponds to the intrinsic curvature perturbation $\Psi$.

where

$$\zeta^B(\boldsymbol{x}) \equiv \int_{k\ll k_1,k_2} \frac{d^3k}{(2\pi)^3}\zeta_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2.91}$$

The background perturbation $\zeta^B$ is small, so it makes sense to expand the correlation function in a power series about $\zeta^B$ and keep only the first term:

$$\langle\zeta^2\rangle_{\zeta^B}(\boldsymbol{x},\Delta\boldsymbol{x}) = \langle\zeta^2\rangle_0(\Delta x) + \int d^3k\zeta^B_{\boldsymbol{k}}\left(\left.\frac{\delta}{\delta\zeta_{\boldsymbol{k}}}\right|_{\zeta^B=0}\langle\zeta^2\rangle_{\zeta^B}(\boldsymbol{x},\Delta\boldsymbol{x})\right) + \cdots, \tag{2.92}$$

where $\boldsymbol{x} \equiv (\boldsymbol{x}_1+\boldsymbol{x}_2)/2$ and $\Delta\boldsymbol{x} \equiv \boldsymbol{x}_2-\boldsymbol{x}_1$. For $\boldsymbol{k}_3$ small enough, we can perform a change of coordinates $\boldsymbol{x} \to \boldsymbol{x}' = e^{\zeta^B(\boldsymbol{x})}\boldsymbol{x}$ to put (2.90) in the form of the unperturbed FLRW metric. In these new coordinates, the background is unperturbed so

$$\langle\zeta^2\rangle_{\zeta^B}(\boldsymbol{x},\Delta\boldsymbol{x}) \approx \langle\zeta^2\rangle_0(|\boldsymbol{x}'_2-\boldsymbol{x}'_1|) \approx \langle\zeta^2\rangle_0(e^{\zeta^B(\boldsymbol{x})}\Delta x). \tag{2.93}$$

Therefore

$$\left.\frac{\delta}{\delta\zeta_{\boldsymbol{k}}}\right|_{\zeta^B=0}\langle\zeta^2\rangle_{\zeta^B}(\boldsymbol{x},\Delta\boldsymbol{x}) = \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{(2\pi)^3}\frac{d[\langle\zeta^2\rangle_0(\Delta x)]}{d\ln\Delta x}. \tag{2.94}$$

Substituting this into (2.92), moving to Fourier space and correlating with $\zeta_{\boldsymbol{k}_3}$, we find

$$\begin{aligned}
\langle\zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\rangle &= \langle\langle\zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\rangle_{\zeta_{\boldsymbol{k}_3}}\zeta_{\boldsymbol{k}_3}\rangle && (2.95)\\
&= (2\pi)^3\delta^{(3)}(\sum_i \boldsymbol{k}_i)P(k_3)FT\left[\frac{d\langle\zeta^2\rangle_0}{d\ln\Delta x}\right](k_S) && (2.96)\\
&= -(2\pi)^3\delta^{(3)}(\sum_i \boldsymbol{k}_i)P(k_3)\frac{1}{k_S^3}\frac{d[k_S^3P(k_S)]}{d\ln k_S} && (2.97)\\
&= (2\pi)^3\delta^{(3)}(\sum_i \boldsymbol{k}_i)P(k_1)P(k_3)(1-n_S), && (2.98)
\end{aligned}$$

where $\boldsymbol{k}_s \equiv (\boldsymbol{k}_1-\boldsymbol{k}_2)/2 \approx \boldsymbol{k}_1$ and we have used the fact that $n_S - 1 \equiv \frac{d\ln[k^3P(k)]}{d\ln k}$. This yields the result of [23].

We have seen in (2.86) that $f_{NL}$ parametrizes the part of the bispectrum that has the form

$$\langle\zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\sum_i k_i)\frac{6}{5}f_{NL}[P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_2)P_\zeta(k_3) + P_\zeta(k_3)P_\zeta(k_1)]. \tag{2.99}$$

Since we measure that $P_\zeta(k) \sim k^{-3}$ [68], we can see that the bispectrum has a "squeezed" shape, namely it peaks at the squeezed limit, and its squeezed form is

$$\langle\zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\rangle_{k_3\ll k_1,k_2} = (2\pi)^3\delta^{(3)}(\sum_i k_i)\frac{12}{5}f_{NL}P_\zeta(k_1)P_\zeta(k_3). \tag{2.100}$$

Confronting the results (2.98) and (2.100) we obtain that for any single-field inflation model

$$f_{NL} = \frac{5}{12}(1-n_S). \tag{2.101}$$

Since $1 - n_S = 0.037$ [68], every single model should produce a level of non-Gaussianity $f_{NL} \simeq 0.02$ during inflation.

## 2.8 Enhancement of non-Gaussianity after inflation

From the study of the late-time evolution of the cosmological perturbations it was found that a large non-linearity is generated by the gravitational dynamics from the original inflationary quantum fluctuations. This leads to a significant enhancement of the tiny intrinsic non-Gaussianity produced during inflation in single-field slow-roll models. We review a general discussion of this argument as given in [70].

As we have seen previously, we can expand $\Delta T$ in spherical harmonics

$$\Delta T(\hat{\boldsymbol{n}}) = \sum_{lm} a_{lm}Y_{lm}(\hat{\boldsymbol{n}}), \tag{2.102}$$

where $\hat{\boldsymbol{n}}$ denotes the direction of observation. Given the form of the gravitational potential $\Phi$, one can calculate the harmonic coefficients

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \Phi_{\rm Pr}(\boldsymbol{k}) g_{Tl}(k) Y^*_{lm}(\boldsymbol{k}), \tag{2.103}$$

where $g_{Tl}$ is the radiation transfer function and the index "Pr" stands for primordial, by which we mean $\Phi_{\rm Pr} = \frac{3}{5}\zeta$. So far, we have assumed that one can use the formula (2.103) to convert the primordial curvature perturbation to the temperature anisotropy. However this equation is valid only for linear theory. Since any non-linear effect can produce non-Gaussianity, one has to study the consequence of various non-linear effects.

The origin of the linear radiation transfer function is the linearized Boltzmann equation

$$\frac{\partial \Delta^{(1)}}{\partial \eta} + ik\mu\Delta^{(1)} + \sigma_T n_e a \Delta^{(1)} = S^{(1)}(k, \mu, \eta), \tag{2.104}$$

where $\eta$ is the conformal time, $\mu \equiv \boldsymbol{k} \cdot \hat{\boldsymbol{n}}$, $\Delta^{(1)} \equiv 4[\Delta T^{(1)}(k, \mu, \eta)/T]$ is the perturbation in the photon energy density, and $S^{(1)}$ is the linear source function, which depends on the metric perturbations and on the density, velocity, pressure and stress perturbations of matter and radiation and the photon polarization. The second-order Boltzmann equation is written in a similar way,

$$\frac{\partial \Delta^{(2)}}{\partial \eta} + ik\mu\Delta^{(2)} + \sigma_T n_e a \Delta^{(2)} = S^{(2)}(\boldsymbol{k}, \hat{\boldsymbol{n}}, \eta), \tag{2.105}$$

where $\Delta^{(2)} \equiv 8[\Delta T^{(2)}(\boldsymbol{k}, \hat{\boldsymbol{n}}, \eta)/T] + 12[\Delta T^{(1)}(\boldsymbol{k}, \hat{\boldsymbol{n}}, \eta)/T]^2$, and $S^{(2)}$ is the second-order source function. Note that at second order the perturbations depend on the directions of $\boldsymbol{k}$ and $\hat{\boldsymbol{n}}$ independently. In this case the second-order $a_{lm}$ is given by [101]

$$\begin{aligned} a^{(2)}_{lm} = \frac{4\pi}{8}(-i)^l \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} \int d^3\boldsymbol{k}'' \delta^D(\boldsymbol{k}' + \boldsymbol{k}'' - \boldsymbol{k}) \Phi^{(1)}_{\rm Pr}(\boldsymbol{k}'') \\ \times \sum_{l'm'} F^{l'm'}_{lm}(\boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{k}) Y^*_{l'm'}(\hat{\boldsymbol{k}}), \end{aligned} \tag{2.106}$$

where $F^{l'm'}_{lm}$ is the second-order radiation transfer function and its form is determined by the second-order source function $S^{(2)}$ in the Boltzmann equation.

The shape of the second order bispectrum, $\langle a^{(1)}_{l_1 m_2} a^{(1)}_{l_2 m_3} a^{(2)}_{l_3 m_3} \rangle$, is given by the second-order radiation transfer function. If the latter vanishes in the squeezed limit, i.e. $F^{l'm'}_{lm}(\boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{k}) \to 0$ for $\boldsymbol{k} \to 0$, then the CMB bispectrum would not peak at the squeezed limit configuration, and thus the resulting $f^{local}_{NL}$ would be small.

The second-order source function is quite complicated, but it can be decomposed into two parts: (a) the terms given by the products of the first-order perturbations, such as $[\Phi^{(1)}]^2$ and (b) the terms given by the "intrinsically second-order terms", such as $\Phi^{(2)}$. This decomposition depends on the gauge choice, so which terms belong to (a) or (b) depends on the gauge. A convenient gauge-choice seems the Newtonian gauge ($B = E = 0$ in (2.39)), chosen by Pitrou, Uzan and Bernardeau [103, 104] and Bartolo, Matarrese and Riotto [14, 15], where the products of the first-order terms only give $|f^{local}_{NL}| < 1$. The intrinsically second order terms are sourced by products of the first-order perturbations and therefore they are created by the late-time evolution of cosmological perturbations, while the terms in (a) are set by the initial conditions. In [104], the authors showed that the terms in (b) give $f^{local}_{NL} \sim 5$ for the Planck data ($l_{max} = 2000$).

# Chapter 3

# The curvaton scenario

As we have seen in the previous chapter, quantum fluctuations of the inflaton field in the simplest inflationary models produce perturbations which are adiabatic, Gaussian and with a slightly red tilted power spectrum. This prediction seems in very good agreement with the observational data. The 7-year WMAP data combined with the latest distance measurements from the Baryon Acoustic Oscillations (BAO) in the distribution of galaxies [110] and the Hubble constant ($H_0$) measurement [111] find $n_s = 0.968 \pm 0.012$ at 68% confidence level [68], confirming the prediction of single-field inflation for the scalar power spectrum. However, there is not yet a very precise measurement of the level of non-Gaussianity of the curvature perturbations. The 7-year WMAP analysis yields the non-linearity parameter $f_{NL}$ for the local form in the following range: $-10 < f_{NL}^{local} < 74$. The prediction for non-Gaussianity of single-field inflation models is in this range. However, other inflationary models which deviate from the standard paradigm and predict a higher level of non-Gaussianity might also fit the data. Since the precision of the observations improves every year, in the near future we will be able to discriminate among the various inflationary models by studying the degree of non-Gaussianity of the perturbations.

A higher level of non-Gaussianity is obtained in inflationary models such as two-field [13, 83, 84, 87], hybrid and multi-brid [3, 9, 10, 42, 59, 99] inflation and in certain modulated and tachyonic (p)re-heating scenarios [36, 40, 60, 129].

In this chapter we study the simplest of these models, which is a model of inflation with two scalar fields, called the *curvaton scenario*. In Section 3.1 we give a basic description of this model describing the generation of curvature perturbations and calculating the level of non-Gaussianity. In Section 3.2 we investigate how to implement the curvaton scenario in supergravity in the context of chaotic inflation and we compute the level of non-Gaussianity which arises in this class of models.

## 3.1 Two-field inflation

It is possible to produce non-Gaussian adiabatic perturbations in models containing several scalar fields. The simplest case is a model with two scalar fields, the inflaton and the *curvaton*. In such a model the power spectrum of perturbations can be non-flat and it can be non-Gaussian. This possibility was first proposed in [81] and then it was significantly developed in [30, 80, 83–85, 91, 92] and many other works.

In this scenario the adiabatic density perturbations are produced after inflation from purely isocurvature curvaton perturbations. The curvaton field is subdominant during inflation and therefore its perturbations are of isocurvature type. The curvature perturbation becomes relevant when the energy density of the curvaton field becomes a significant fraction of the total energy density. This happens when the Hubble parameter drops below the curvaton mass and the curvaton field starts to oscillate around the minimum of its potential, behaving like non-relativistic matter. Afterwards the curvaton decays into thermalized radiation generating an adiabatic perturbation. It is also possible that some residual isocurvature perturbations survive after the curvaton decay. For example in the case where the curvaton, when subdominant, decays into a component of Cold Dark Matter (CDM) which does not thermalize with the existing radiation. We now analyze the generation of curvaton perturbations following [83] and [12].

### 3.1.1 The curvaton field perturbation

During inflation the curvaton field $\sigma$ is supposed to be an almost free scalar field with effective mass $m_\sigma^2 = |V_{\sigma\sigma}| \ll H^2$ and it is supposed to give a negligible contribution to the energy density. It is assumed that the curvaton field does not have any significant coupling with other fields or that the effect of any coupling can be integrated out to give a possibly time-dependent potential $V$. The unperturbed curvaton field satisfies the equation of motion

$$\ddot{\sigma} + 3H\dot{\sigma} + V_{,\sigma} = 0. \tag{3.1}$$

If we expand the curvaton field as $\sigma(t, \boldsymbol{x}) = \sigma(t) + \delta\sigma(t, \boldsymbol{x})$, then the perturbation satisfies the equation

$$\delta\ddot{\sigma} + 3H\delta\dot{\sigma} + V_{,\sigma\sigma}\delta\sigma = 0, \tag{3.2}$$

where we made the first-order approximation $\delta(V_{,\sigma}(t, \boldsymbol{x})) \approx V_{,\sigma\sigma}(t)\delta\sigma(t, \boldsymbol{x})$. The fluctuations $\delta\sigma$ on super-horizon scales are Gaussian with an almost scale-invariant power spectrum:

$$\delta^2_{\delta\sigma} \approx \frac{H_*^2}{4\pi^2}, \tag{3.3}$$

where $*$ denotes the epoch of horizon exit $k = aH$. After the end of inflation, the inflaton energy density is converted into radiation and the curvaton field remains approximately constant until $H^2 \sim m_\sigma^2$. When the Hubble parameter falls below the curvaton mass, the curvaton starts to oscillate around the minimum of its potential. Even if the potential is not quadratic, after a few Hubble times, we can make the approximation $V \approx \frac{1}{2}m_\sigma^2\sigma^2$ and the energy density will be

$$\rho_\sigma(t, \boldsymbol{x}) \approx m_\sigma^2\sigma^2(t, \boldsymbol{x}). \tag{3.4}$$

From Equations (3.1) and (3.2) we see that for a quadratic potential the ratio $\delta\sigma/\sigma$ does not evolve. The perturbation in $\rho_\sigma$ depends on the curvaton field perturbation through both a linear and a quadratic term. Assuming, for the moment, that the linear term dominates, the resulting relative energy density perturbation is

$$\frac{\delta\rho_\sigma}{\rho_\sigma} = 2\left(\frac{\delta\sigma}{\sigma}\right)_*. \tag{3.5}$$

### 3.1.2 The curvature perturbation

Perturbations in the energy density of the curvaton field produce a primordial density perturbation well after the end of inflation. The primordial adiabatic density perturbation is associated with a perturbation in the spatial curvature $\Psi$ and it is characterized by the gauge-invariant variable $\zeta$, which was introduced in the previous chapter, Equation (2.75).

When the Hubble parameter falls below the curvaton mass, the curvaton field starts to oscillate. At this moment, the energy density is dominated by radiation ($\gamma$), which is the result of the decay of the inflaton field. During the curvaton oscillations, $\rho_\sigma \propto a^{-3}$ and $\rho_\gamma \propto a^{-4}$. Therefore the curvaton component of the energy density $\rho_\sigma$ increases with respect to the radiation component $\rho_\gamma$ and the perturbations in the curvaton field are then converted into the curvature perturbation.

To analyze the generation of the curvature perturbation it is convenient to consider the curvature perturbations $\zeta_i$ associated with each individual energy density component:

$$\zeta_i \equiv -\Psi - H\left(\frac{\delta\rho_i}{\dot{\rho}_i}\right). \tag{3.6}$$

In particular, on unperturbed hypersurfaces $\Psi = 0$ (*spatially flat gauge*), $\zeta_\sigma$ is

$$\zeta_\sigma = \frac{1}{3}\frac{\delta\rho_\sigma}{\rho_\sigma}. \tag{3.7}$$

and thus the total curvature perturbation can be written as the weighted sum

$$\zeta = (1-f)\zeta_\gamma + f\zeta_\sigma, \tag{3.8}$$

where $f$ defines the relative contribution of the curvaton field to the total curvature perturbation and it is given by

$$f = \frac{3\rho_\sigma}{4\rho_\gamma + 3\rho_\sigma}. \tag{3.9}$$

In the following we assume the approximation of sudden decay of the curvaton field. Under this assumption, the radiation and the curvaton field satisfy separate conservation equations

$$\dot{\rho}_\gamma = -4H\rho_\gamma, \tag{3.10}$$

$$\dot{\rho}_\sigma = -3H\rho_\sigma, \tag{3.11}$$

and each $\zeta_i$ is constant on super-horizon scales until the curvaton decay. Thus from Equation (3.8) it follows that the evolution of $\zeta$ on these scales is given by

$$\dot{\zeta} = \dot{f}(\zeta_\sigma - \zeta_\gamma) = Hf(1-f)(\zeta_\sigma - \zeta_\gamma). \tag{3.12}$$

The curvaton scenario corresponds to the case where the curvature perturbation in the radiation produced at the end of inflation is negligible, $\zeta_\gamma \approx 0$. In the sudden decay approximation, $\zeta_\gamma$ and $\zeta_\sigma$ both remain constant up until the curvaton decays. Well after the decay of the curvaton, during the matter and radiation eras, the curvature perturbation stays constant on super-horizon scales at a value which is fixed by

$$\zeta \approx f_{dec}\zeta_\sigma, \tag{3.13}$$

where $f_{dec}$ is $f$ at the decay time. Going beyond the sudden decay approximation, we can introduce a parameter $r$ defined in the following way:

$$\zeta = r\zeta_\sigma, \tag{3.14}$$

$$= \frac{r}{3}\frac{\delta\rho_\sigma}{\rho_\sigma}, \tag{3.15}$$

where $\zeta$ is evaluated after the curvaton decay and $\zeta_\sigma$ is evaluated before the curvaton decay. In the case where the curvaton completely dominates the energy density before it decays, $r = 1$. In this limit the sudden decay approximation becomes exact. In the case where the curvaton does not dominate, numerical studies performed in [88] show that

$$r \approx \left(\frac{\rho_\sigma}{\rho_{tot}}\right)_{dec}. \tag{3.16}$$

The prediction of the curvaton model for the spectrum of the curvature perturbation is

$$\delta_\zeta = \frac{2}{3} r \delta_{\delta\sigma/\sigma}. \tag{3.17}$$

The COBE measurements of the CMB anisotropy require $\delta_\zeta(\mathrm{COBE}) = 4.8 \times 10^{-5}$. Therefore, if the curvaton dominates the energy density before it decays ($r = 1$), this implies that

$$\delta_{\delta\sigma/\sigma} = 7.2 \times 10^{-5}. \tag{3.18}$$

### 3.1.3 Non-Gaussianity of the curvature perturbation

From Equation (3.4) and (3.15) we can see that the curvature perturbation depends on the curvaton field perturbations through both a linear and a quadratic term. The linear term gives a Gaussian contribution to the curvature perturbation, but if the quadratic term in the energy perturbation is not negligible, then the curvature perturbation will have a non-Gaussian component.

The level of non-Gaussianity is encoded in the non-linearity parameter $f_{NL}$. As we have seen previously, a phenomenological way of parametrizing the level of non-Gaussianity is to expand the fully nonlinear primordial Bardeen gravitational potential $\Phi$ in powers of the linear gravitational potential $\Phi_L$,

$$\Phi = \Phi_L + f_{NL}\Phi_L^2. \tag{3.19}$$

We have seen that the relation between $\Phi$ and $\zeta$ during the matter era on super-horizon scales is $\Phi = \frac{3}{5}\zeta$, thus

$$\Phi = \frac{r}{5}\frac{\delta\rho_\sigma}{\rho_\sigma}. \tag{3.20}$$

Equation (3.4) gives

$$\frac{\delta\rho_\sigma}{\rho_\sigma} = 2\frac{\delta\sigma}{\sigma} + \frac{(\delta\sigma)^2}{\sigma^2}. \tag{3.21}$$

Therefore, using Equations (3.19), (3.20) and (3.21), we obtain the prediction of the curvaton scenario for the level of non-Gaussianity:

$$f_{NL} = \frac{5}{4r}. \tag{3.22}$$

We should note that in order to make this estimate we assumed first-order cosmological perturbation theory. The validity of the estimate (3.22) requires that the curvaton contributes only a small fraction of the energy density before it decays. If the curvaton dominates the energy density before its decay, the non-Gaussianity calculated at linear order is lost in the noise of the unknown second-order corrections to cosmological perturbation theory.

We have seen that the curvaton scenario can be consistent with the observations. In fact the density fluctuations generated in such a scenario are almost scale-invariant and the level of non-Gaussianity can be quite significant. In order to generate large non-Gaussianity with the curvaton, it is necessary that the energy density of the curvaton at the time of its decay is much smaller than that of the dominant component of the Universe (which is expected to be from the inflaton).

## 3.2 The curvaton scenario and supergravity

The success of the inflationary paradigm in solving the main puzzles of the early Universe is outstanding. Nontheless we still do not have a derivation from first principles of a theory of inflation. In fact we lack a natural way of identifying the fields involved with fundamental fields in particle physics.

*Supersymmetry* is widely discussed as the most interesting candidate for the physics beyond the standard model. It may solve the *hierarchy problem*, namely the discrepancy between the experimentally anticipated order of magnitude for the Higgs boson mass and its theoretical expectation. It also allows for the unification of the weak interaction, the strong interaction and electromagnetism, in the sense that the values of the three coupling constants agree at a certain energy scale only in a supersymmetric version of the standard model.

In the previous chapter we have seen that in the simplest inflationary model inflation is due to the potential energy of a scalar field. Such a potential must be relatively flat in order to guarantee a long duration of inflation and a small deviation from scale invariance of primordial density fluctuations. However, the flatness of the scalar potential can be easily destroyed by radiative corrections. This problem can be solved by supersymmetry. When combined with gravity, supersymmetry must be a local symmetry. Such a supersymmetric version of gravity is called *supergravity*, and its action is uniquely specified by the choice of two functions, the Kähler potential $\mathcal{K}$ and the superpotential $W$. For review see [125]. Therefore it is very natural to consider inflation in the context of supergravity.

### 3.2.1 Chaotic inflation in supergravity

Among various inflationary models, *chaotic inflation* [76] seems very attractive since it is very simple and it does not suffer from any initial condition problem. However for many years, it seemed very difficult to realize chaotic inflation in supergravity. This issue was solved by Kawasaki, Yamaguchi and Yanagida in [66]. In the following we review their argument.

The main problem to overcome in order to realize simultaneously supergravity and chaotic inflation is the fact that the minimal supergravity potential has an exponential factor, $\exp\left(\phi_i^*\phi_i/M_P^2\right)$, which prevents any scalar field $\phi_i$ from having values larger than $M_P \simeq 10^{19}$ GeV. However in chaotic inflation, the inflaton field $\phi$ must have a value larger than $M_P$ in order to cause inflation.

In [66] the problem was solved assuming that the form of the Kähler potential is determined by a symmetry. With this symmetry the inflaton field is allowed to have values larger than $M_P$ and hence it can cause inflation. The authors assumed that the Kähler potential $\mathcal{K}(\Phi, \Phi^*)$ is invariant under the shift symmetry of $\Phi$,

$$\Phi \rightarrow \Phi + iCM_P, \tag{3.23}$$

where $C$ is a dimensionless parameter. Therefore the Kähler potential will be a function of $\Phi + \Phi^*$, $\mathcal{K}(\Phi, \Phi^*) = \mathcal{K}(\Phi + \Phi^*)$. The imaginary part of the field $\Phi$ is canceled out in the Kähler potential and therefore it is allowed to have values larger than $M_P$. For this reason we can identify the imaginary part of $\Phi$ with the inflaton field $\phi$. However, as long as the shift symmetry is exact, the inflaton field never has a potential and it never causes inflation. Thus it is necessary to introduce a small breaking term for $\Phi$ in the superpotential

$$W = mS\Phi, \tag{3.24}$$

where $S(x, \theta)$ is a new superfield. The potential is given by

$$V = e^{\mathcal{K}} \left[ \left( \frac{\partial^2 \mathcal{K}}{\partial \Phi \partial \Phi^*} \right)^{-1} D_\Phi W D_{\Phi^*} W^* - 3|W|^2 \right], \tag{3.25}$$

where

$$D_\Phi W = \frac{\partial W}{\partial \Phi} + \frac{\partial \mathcal{K}(\Phi + \Phi^*)}{\partial \Phi} W, \tag{3.26}$$

here $\Phi$ is the scalar component of the superfield $\Phi$ and we set $M_P = 1$. We should notice that this model possesses $U(1)_R$ symmetry under which

$$S(\theta) \to e^{-2i\alpha} S(\theta e^{i\alpha}), \tag{3.27}$$

$$\Phi(\theta) \to \Phi(\theta e^{i\alpha}). \tag{3.28}$$

We consider that the small parameter $m$ is originated from small breaking of the shift symmetry in a more fundamental theory and as long as $m \ll \mathcal{O}(1)$, the corrections from the breaking term (3.24) to the Kähler potential are negligibly small. Then we assume that the Kähler potential has the shift symmetry (3.23) and the $U(1)_R \times Z_2$ symmetry neglecting the breaking effects,

$$\mathcal{K}(\Phi, \Phi^*, S, S^*) = \mathcal{K}[(\Phi + \Phi^*)^2, SS^*]. \tag{3.29}$$

In the following analysis we take

$$\mathcal{K} = \frac{1}{2}(\Phi + \Phi^*)^2 + SS^* + ... \tag{3.30}$$

The Lagrangian density is now given by

$$L(\Phi, S) = \partial_\mu \Phi \partial^\mu \Phi^* + \partial_\mu S \partial^\mu S^* - V(\Phi, S), \tag{3.31}$$

with potential

$$V(\Phi, S) = m^2 e^{\mathcal{K}} \left[ |\Phi|^2 (1 + |S|^4) + |S|^2 \{1 - |\Phi|^2 + (\Phi + \Phi^*)^2 (1 + |\Phi|^2)\} \right], \tag{3.32}$$

where we have neglected higher order terms in the Kähler potential and $S$ denotes the scalar component of the superfield $S$. After decomposing the complex scalar field $\Phi$ as $\Phi = (\eta + i\phi)/\sqrt{2}$, the Lagrangian density takes the form

$$L(\eta, \phi, S) = \frac{1}{2}\partial_\mu \eta \partial^\mu \eta + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \partial_\mu S \partial^\mu S^* - V(\eta, \phi, S), \tag{3.33}$$

with

$$\begin{aligned} V(\eta, \phi, S) &= m^2 \exp\left(\eta^2 + |S|^2\right) \\ &\times \left[\frac{1}{2}(\eta^2+\phi^2)(1+|S|^4) + |S|^2 \left\{1 - \frac{1}{2}(\eta^2+\phi^2) + 2\eta^2\left(1 + \frac{1}{2}(\eta^2+\phi^2)\right)\right\}\right]. \end{aligned} \tag{3.34}$$

Because of the presence of the factor $e^{\mathcal{K}}$, we have that $|\eta|, |S| \ll \mathcal{O}(1)$. $\phi$ can take values much larger than $\mathcal{O}(1)$ since $e^{\mathcal{K}}$ does not contain $\phi$. For $|\eta|, |S| \ll \mathcal{O}(1)$, the potential becomes

$$V(\eta, \phi, S) \simeq \frac{1}{2}m^2\phi^2(1+\eta^2) + m^2|S|^2. \tag{3.35}$$

The initial value of $\phi_i$ is determined so that $V(\phi_i) \sim \frac{1}{2}m^2\phi_i^2 \sim 1$, thus we have $\phi_i \sim m^{-1} \gg 1$. For such a large value of $\phi$ the effective mass of $\eta$ becomes much larger than $m$, so $\eta$ is quickly stabilized at $\eta = 0$. The field $S$ has a light mass and slowly rolls down to $S = 0$. The potential (3.35) becomes

$$V(\eta, \phi) \simeq \frac{1}{2}m^2\phi^2 + m^2|S|^2. \tag{3.36}$$

Since $\phi \gg 1$ and $|S| < 1$, the field $\phi$ dominates the potential and chaotic inflation takes place.

Therefore we see that chaotic inflation naturally takes place if we assume that the Kähler potential is invariant under a shift symmetry of the inflaton field and introduce a small breaking term of this shift symmetry.

### 3.2.2 Supercurvaton

In this section we will follow our paper [26]. One of the main reasons to introduce the curvaton scenario was to obtain a realistic mechanism of generation of non-Gaussian adiabatic perturbations of metric. Since that time, many interesting curvaton models were proposed. However, it would be nice to have a curvaton model which would be as simple as the basic chaotic inflation scenario with the potential $m^2\phi^2/2$ [76]. It would be good also to find a natural implementation of this scenario in the context of supergravity.

As we have seen in Section 3.2.1, it is possible to implement chaotic inflation in supergravity. The authors of [66] proposed a very simple model describing two fields, $S$ and $\Phi$, with the superpotential

$$W = mS\Phi, \tag{3.37}$$

and Kähler potential

$$\mathcal{K} = SS^* - \frac{1}{2}(\Phi - \Phi^*)^2. \tag{3.38}$$

Note that the Kähler potential does not depend on the phase of the field $S$ and on the real part of the field $\Phi$. Therefore it will be convenient for us to represent the fields $S$ and $\Phi$ as $S = \sigma\, e^{i\theta}/\sqrt{2}$ and $\Phi = (\phi + i\chi)/\sqrt{2}$. The field $\phi$ plays the role of the inflaton field, with the quadratic potential, as in the simplest version of the chaotic inflation scenario [76]:

$$V(\phi) = 3H^2 = \frac{m^2}{2}\phi^2, \tag{3.39}$$

where $H$ is the Hubble constant during inflation. Near the inflationary trajectory with $S = 0$, the mass squared of the imaginary part of the field $\Phi$ is $m_\chi^2 = 6H^2 + m^2$. Thus during inflation $m_\chi^2 > 6H^2$, and therefore the imaginary part of the field $\Phi$ is stabilized at $\mathrm{Im}\,\Phi = 0$. No perturbations of this field are generated.

Both components of the field $S$ may remain light during inflation, and therefore inflationary perturbations of these fields can be generated [25]. Since the potential does not depend on the field $\theta$, we will ignore fluctuations of this field in our study of the curvaton perturbations. The potential of the fields $\phi$, $\sigma$ at $\chi = 0$ is

$$V(\phi, \sigma) = \frac{m^2}{2} e^{\sigma^2/2} \left[\phi^2 + \sigma^2 + \frac{\phi^2}{4}\sigma^2(\sigma^2 - 2)\right]. \tag{3.40}$$

For $\sigma \ll 1$ one has

$$V(\phi,\sigma) = \frac{m^2\phi^2}{2} + \frac{m^2\sigma^2}{2} + \frac{m^2\phi^2\sigma^4}{16}. \tag{3.41}$$

The effective mass squared of the field $\sigma$ at $\sigma \ll 1$ is given by

$$m_\sigma^2 = V_{\sigma\sigma} = m^2 + \frac{3}{4}m^2\phi^2\sigma^2 = m^2 + \frac{9}{2}H^2\sigma^2, \tag{3.42}$$

where $V_{\sigma\sigma}$ means second partial derivative of $V$ with respect to $\sigma$. One can easily see that $m_\sigma^2 = m^2$ for $\phi\sigma \ll 1$. During inflation $m^2 \ll H^2$, and therefore inflationary perturbations of the field $\sigma$ can be generated. At $\phi\sigma \gtrsim 1$, the effective mass squared of the field $\sigma$ is dominated by the term $\frac{3}{4}m^2\phi^2\sigma^2 = \frac{9}{2}H^2\sigma^2 > m^2$. For $\sigma \ll 1$ one still has $m_\sigma^2 \ll H^2$, so the perturbations of the field $\sigma$ are generated in this regime as well. However, at $\sigma \gtrsim 1$ the potential becomes exponentially steep, and $m_\sigma^2 \gg H^2$. Therefore inflationary fluctuations of this field are generated only for $\sigma \lesssim 1$. This is a very important advantage of the curvaton scenario in supergravity: the steepness of the curvaton potential at $\sigma \gtrsim 1$ protects us from extremely large perturbations of the curvaton field which otherwise could be produced during eternal inflation in this scenario [77, 80].

If one does not take into account the curvaton fluctuations in this scenario and studies only the usual inflaton fluctuations [8, 53, 56, 95, 96, 115], then the COBE normalization requires $m \sim 6\times10^{-6}$, in the system of units $M_P = 1$ [76, 78, 79, 93]. Thus, the mass of the inflaton field must be somewhat smaller than $6\times10^{-6}$ if we want to add the curvaton fluctuations to the inflaton fluctuations.

Recently the supergravity model described above was substantially generalized in [63, 64]. The generalized scenario describes a theory with a superpotential

$$W = Sf(\Phi), \tag{3.43}$$

where $f(\Phi)$ is an arbitrary real holomorphic function. The Kähler potential in this class of models may take several different functional forms, e.g.

$$\mathcal{K} = S\bar{S} - \frac{1}{2}(\Phi-\bar{\Phi})^2 - \frac{\alpha}{12}(S\bar{S})^2. \tag{3.44}$$

In this theory, the inflaton potential is given by

$$V(\phi) = f^2(\phi/\sqrt{2}) \tag{3.45}$$

and the mass of the field $\sigma$ is

$$m_\sigma^2 = \alpha H^2 + (f'(\phi/\sqrt{2}))^2. \tag{3.46}$$

In this class of models, one can implement chaotic inflation in supergravity, with an *arbitrary* shape of the inflaton potential $V(\phi)$. In all of these models one has $H^2 = f^2(\phi/\sqrt{2})/3$. The term $(f'(\phi/\sqrt{2}))^2$ is equal to $3H^2\epsilon$, where $\epsilon \ll 1$ is the slow roll parameter. For $\alpha \gtrsim 1$ one has $m_\sigma^2 \gtrsim H^2$. In this case no curvaton perturbations are produced, so all standard predictions of the single-field inflaton scenario remain intact.

On the other hand, in models with $\alpha \ll 1$ one has $m_\sigma^2 \ll H^2$ during inflation, which means that quantum fluctuations of the field $\sigma$ are generated during inflation [63, 64].

Thus we have a broad class of models of chaotic inflation where the curvaton scenario can be realized. One can further generalize this scenario by adding terms $\sim S^3$ to the superpotential, and by using other versions of the Kähler potential, as long as the Kähler potential has certain properties described in [63, 64]. The requirements which are necessary for the existence of the light curvaton fields in this class of models can be formulated in an invariant way in terms of the curvature of the Kähler geometry. In particular, the parameter $\alpha$ is related to the curvature of the Kähler manifold [64]. The field $\sigma$ itself has an interesting interpretation from the point of view of supergravity: it is the scalar component $\sigma$ of the goldstino multiplet. Because of the generality and simplicity of this scenario and because of its supergravity origin, one may call it the *supercurvaton scenario*.

Here we will concentrate on the simplest model (3.37), (3.38), but with an additional term $-\frac{\alpha}{12}(S\bar{S})^2$ in the Kähler potential, as in Equation (3.44). In this model the curvaton mass squared along the inflationary trajectory with $\sigma = 0$ is given by

$$m_\sigma^2 = m^2 + \alpha H^2, \tag{3.47}$$

and in a more general case $0 < \sigma \ll 1$ the effective mass squared of the field $\sigma$ is

$$m_\sigma^2 = m^2 + \alpha H^2 + \frac{9}{2}H^2\sigma^2 = m^2 + \frac{\alpha}{6}m^2\phi^2 + \frac{3}{4}m^2\phi^2\sigma^2. \tag{3.48}$$

#### 3.2.2.1 Curvaton perturbations and non-Gaussianity

During inflation, the curvaton perturbations are produced. An average amplitude of perturbations produced during each Hubble time $H^{-1}$ is given by $\delta\sigma = \frac{H}{2\pi}$. Then these fluctuations are stretched, overlap with each other, and eventually produce a classical curvaton field $\sigma$ which looks relatively homogeneous in the observable part of the Universe, but may take different values in other parts of the Universe [80]. The amplitude of the perturbations of density of the curvaton field with a quadratic potential is given by $\delta\rho_\sigma/\rho_\sigma \sim 2\delta\sigma/\sigma$. However, the total energy density of matter at the moment when the curvaton field decays may be greater than the energy of the classical field $\sigma$. This may happen, for example, if the decay of the inflaton field during reheating produces many curvaton particles [80]. Therefore the relative perturbation of density will be given by

$$\frac{\delta\rho_\sigma}{\rho} \sim \frac{2r\delta\sigma}{\sigma}, \tag{3.49}$$

where $r = \rho_\sigma/\rho$ at the time of the curvaton decay. According to [83], these perturbations will match the COBE normalization of the spectrum for

$$r\frac{\delta\sigma}{\sigma} \sim 7 \times 10^{-5}. \tag{3.50}$$

As we have seen in the previous section, these perturbations are non-Gaussian, with the amplitude of local non-Gaussianity given by [83]

$$f_{\rm NL} = \frac{5}{4r}. \tag{3.51}$$

Our goal will be to find a typical value of $\sigma$ in some of the simplest supergravity models described above, calculate $\delta\sigma$, find the value of $r$ required to satisfy

Equation (3.50), and finally determine $f_{\rm NL}$. The most complicated part of this program is finding a typical value of $\sigma$.

#### 3.2.2.2 Stochastic approach

We will begin our study with an investigation of the behavior of the distribution of the fluctuations of the curvaton field $\sigma$ with a simple quadratic potential $m_\sigma^2\sigma^2/2$. This approach will allow us to describe the case when $m_\sigma^2 = m^2 + \alpha H^2$, but not the more general situation when $m_\sigma^2$ depends on $\sigma$ as in (3.48), which will be discussed separately later.

During inflation, the long-wavelength distribution of this field generated at the early stages of inflation behaves as a nearly homogeneous classical field, which satisfies the equation

$$3H\dot\sigma + V_\sigma = 0, \tag{3.52}$$

or, equivalently,

$$\frac{d\sigma^2}{dt} = -\frac{2V_\sigma\,\sigma}{3H}. \tag{3.53}$$

However, each time interval $H^{-1}$ new fluctuations of the scalar field are generated, with an average amplitude squared

$$\langle\delta\sigma^2\rangle = \frac{H^2}{4\pi^2}. \tag{3.54}$$

The wavelength of these fluctuations is rapidly stretched by inflation. This effect increases the average value of the square of the classical field $\sigma$ in a process similar to Brownian motion. As a result, the square of the field $\sigma$ at any given point with inflationary fluctuations taken into account changes, in average, with the speed which differs from the predictions of the classical equation of motion by $\frac{H^3}{4\pi^2}$:

$$\frac{d\sigma^2}{dt} = -\frac{2V_\sigma\,\sigma}{3H} + \frac{H^3}{4\pi^2}. \tag{3.55}$$

Using $3H\dot\phi = -V_\phi$, one can rewrite this equation as

$$\frac{d\sigma^2}{d\phi} = \frac{2V_\sigma\,\sigma}{V_\phi} - \frac{V^2}{12\pi^2 V_\phi}. \tag{3.56}$$

Its solution with the initial condition $\sigma(\phi_i) = 0$ for $m_\sigma^2 = m^2 + \alpha H^2$ is given by

$$\sigma^2(\phi) = \frac{1}{12\pi^2} \int\limits_{\phi}^{\phi_i} \frac{V^2(\tilde{\phi})}{V'(\tilde{\phi})} \exp\left( -2 \int\limits_{\phi}^{\tilde{\phi}} \frac{m_\sigma^2}{V'(\bar{\phi})} d\bar{\phi} \right) d\tilde{\phi}, \tag{3.57}$$

where $\phi_i$ is the initial value of the inflaton field.

If inflation continued for much longer time than 60 e-foldings, as we will assume here, the main contribution to $\sigma$ is given by perturbations produced at the very early stages of inflation. Such fluctuations look almost absolutely homogeneous on the scale of the observable part of our Universe, so our calculations give us a typical value of the classical field $\sigma$ inside the observable part of our Universe. However, in different parts of the Universe, the field $\sigma$ may be significantly smaller or greater than its "typical" value calculated above. As a result, the amplitude of the curvaton perturbations is not a constant, but it varies in space [80]. The same is true for the degree of non-Gaussianity $f_{\rm NL}$, see Section 3.2.2.8.

Therefore, to be precise, one should distinguish between the average amplitude of the field $\sigma$ calculated above, when the averaging it taken all over the Universe, and the local value of the field $\sigma$ in each horizon-size part of the Universe. We will make this distinction in Section 3.2.2.8, where we will make a slight change of notation and call the value of the curvaton field averaged over the whole Universe $\bar{\sigma}$, reserving the letter $\sigma$ for the average value of the curvaton field in the horizon-size part of the Universe. However, in the main part of this work we will not distinguish between $\sigma$ and $\bar{\sigma}$. This means, in particular, that when we will calculate $f_{\rm NL}(\sigma)$, our results will in fact describe the value of this parameter for $\sigma = \bar{\sigma}$, i.e. the value of $f_{\rm NL}$ for an average value of $\sigma$, all over the Universe. In Section 3.2.2.8 we will show, however, that the value of $f_{\rm NL}$ for an average value of $\sigma$ can be significantly different from the average value of $f_{\rm NL}$; the order of averaging in certain cases can be very important. One should take this effect into account when making predictions of the non-Gaussianity in each particular curvaton scenario.

We should note also that in general the curvaton field may not be equal to zero at the beginning of chaotic inflation, so one may also consider a possibility that initially $\langle\sigma^2\rangle(\phi_i)$ was very large. In this respect, the supergravity model which we are going to study provides an important simplification: The curvaton field initially cannot be much larger than $\mathcal{O}(1)$ because of the exponential steepness of the potential at $\sigma \gtrsim 1$. Also, the effective mass term $\frac{3}{4}m^2\phi^2\sigma^2$ in the supergravity potential (3.40) rapidly reduces the initial value of the field $\sigma$, thus making quantum fluctuations generated during inflation more important than the initial value of the classical field $\sigma$. We will study these issues in the following, starting from the simple toy model with $m_\sigma^2 = m^2$ and ending up with the model with $m_\sigma^2 = m^2 + \frac{\alpha}{6}m^2\phi^2 + \frac{3}{4}m^2\phi^2\sigma^2$. As we will see, in all these cases the final result does not depend on the initial distribution of the curvaton field if inflation lasts long enough.

#### 3.2.2.3 A toy model with $m_\sigma^2 = m^2$

In this section we will study the distribution of the curvaton field with the mass $m_\sigma^2 = m^2$ during inflation driven by the massive inflaton field with potential $V = \frac{1}{2}m^2\phi^2$. In this case Equation (3.57) implies that the classical scalar field $\sigma$ which is nearly homogeneous on the scale of the horizon has a typical amplitude

$$\sigma(\phi) = \frac{m\,\phi\,\phi_i}{4\pi\sqrt{6}}. \tag{3.58}$$

Meanwhile the amplitude of fluctuations of $\sigma$ generated at that time is

$$\delta\sigma \sim \frac{H}{2\pi} = \frac{m\phi}{2\pi\sqrt{6}}. \tag{3.59}$$

During the subsequent evolution of the Universe, $\sigma$ and $\delta\sigma$ both decrease in the same way, and therefore at the end of inflation the curvaton perturbations have flat spectrum with the amplitude

$$\frac{\delta\sigma}{\sigma} = \frac{2}{\phi_i}. \tag{3.60}$$

As we mentioned above the amplitude of the perturbations must be normalized as

$$r\frac{\delta\sigma}{\sigma} \simeq \frac{2r}{\phi_i} \sim 7\times 10^{-5}, \tag{3.61}$$

and hence

$$f_{\rm NL} = \frac{5}{4r} \sim \frac{3.5 \times 10^4}{\phi_i}. \tag{3.62}$$

This means that the degree of non-Gaussianity depends on the initial value of the inflaton field. Unless this field is very large, $f_{\rm NL}$ may be extremely large.

However, in the supergravity models which we study in this work the approach developed above is valid only if inflation was short enough, that is, $\phi_i \ll m^{-1/3}$ and for large values of $\phi_i$ one cannot ignore the supergravity correction to the mass in Equation (3.42).

#### 3.2.2.4 $m_\sigma^2 = m^2 + \frac{9}{2}H^2\sigma^2$

In the previous section we made a simplifying assumption that the curvaton mass does not depend on $\sigma$, which allowed us to use Equation (3.57). However, as one can see from (3.40), in the supergravity model (3.37), (3.38) the curvaton mass does depend on $\sigma$ in a rather complicated way. The leading correction to the curvaton mass squared $m^2$ is given by $\frac{3}{4}m^2\phi^2\sigma^2 = \frac{9}{2}H^2\sigma^2$ and it becomes dominant for $\phi\sigma \gtrsim 1$.

To find out how it will change the final result one has to solve Equation (3.56) for

$$V = \frac{m^2\phi^2}{2} + \frac{m^2\sigma^2}{2} + \frac{m^2\phi^2\sigma^4}{16}, \tag{3.63}$$

which takes in this case the form

$$y' = \frac{y}{x} + \frac{y^2}{4} - bx, \tag{3.64}$$

where $x = \phi^2$, $y = \sigma^2$, and $b = \frac{m^2}{96\pi^2}$.

The general solution of this equation can be expressed in terms of Airy functions,

$$y(x) = -(2b)^{2/3}x\,\frac{{\rm Ai}(z) - c\,{\rm Bi}(z)}{{\rm Ai}'(z) - c\,{\rm Bi}'(z)}, \tag{3.65}$$

where $z = 2^{-2/3}b^{1/3}x$ and $c$ is a small constant which should be chosen in such a way that $y(x_i) = 0$.

Suppose first that the initial value of the field $\phi$ is much higher than $m^{-1/3}$, i.e. $z \gg 1$. One can check that in this case one should take $c \ll 1$ to have

$y(x_i) = 0$. Inflation ends at $\phi \sim 1$, when $z \ll 1$. In this limit, all functions are $\mathcal{O}(1)$. Therefore the functions $\mathrm{Bi}(z)$ drop out from the final expression because of the small coefficient $c$, $\mathrm{Ai}(z) \approx \mathrm{Ai}(0) = 3^{-2/3}\,\Gamma^{-1}(2/3)$, $\mathrm{Ai}'(z) \approx \mathrm{Ai}'^{-1/3}\,\Gamma^{-1}(1/3)$. As a result,

$$y(x) \approx -\frac{(2b)^{2/3}\,x\,\Gamma(1/3)}{3^{1/3}\,\Gamma(2/3)}\ . \tag{3.66}$$

Expressing everything in terms of the original fields $\phi$ and $\sigma$, we find

$$\sigma(\phi) \approx -\frac{m^{2/3}\,\phi}{2^{4/3}\sqrt{3}\pi^{2/3}}\frac{\Gamma(1/3)}{\Gamma(2/3)} \approx 0.15\ m^{2/3}\,\phi. \tag{3.67}$$

This yields

$$\frac{\delta\sigma}{\sigma} \sim 0.4\,m^{1/3}. \tag{3.68}$$

The COBE normalization requires $rm^{1/3} \sim 7\times 10^{-5}$. Therefore, for $m \sim 10^{-7}$ we have $r \sim 0.04$ and $f_{\mathrm{NL}} \sim 30$.

If, on the other hand, the initial value of field $\phi$ is much smaller than $m^{-1/3}$, then the final result looses its universality and become sensitive to $\phi_i$. In this case, one can either use the analytical solution above, with different initial conditions, or simply use the results of the previous section (one can see that in this case $\phi\sigma \ll 1$, and hence the results of Section 3.2.2.3 are valid).

Note that in our calculation of $f_{\mathrm{NL}}$ we used Equation (3.51), which was obtained in [83] under the assumption that the curvaton potential is purely quadratic. Meanwhile in our case the curvaton potential contains the quadratic term $\frac{m^2\sigma^2}{2}$ as well as the quartic term $\frac{m^2\phi^2\sigma^4}{16}$, see (3.63). This could lead to some corrections to Equation (3.51) [41, 113]. Fortunately, one can show that during the last 60 e-foldings of inflation in our model the quartic term is vanishingly small as compared to the quadratic term. That is why one can use the simple Equation (3.51) for the calculation of $f_{\mathrm{NL}}$.

#### 3.2.2.5 $m_\sigma^2 = m^2 + \alpha H^2,\ \ \alpha > 0$

Now we will consider the case when the mass of the curvaton field is given by

$$m_\sigma^2 = \alpha H^2 + m^2 = m^2\left(\frac{\alpha\,\phi^2}{6} + 1\right), \tag{3.69}$$

where we have ignored the correction $\frac{3}{4}m^2\phi^2\sigma^2$, which will be taken into account in Section 3.2.2.6.

To study this case we consider separately the evolution of perturbations at $\alpha\phi^2/6 > 1$ and $\alpha\phi^2/6 < 1$ assuming that during the last 60 e-folds of inflation the condition $\alpha\phi^2/6 < 1$ is satisfied, which means that $\alpha \lesssim 1/40$. Thus during the last 60 e-folds, $m_\sigma^2 \approx m^2$, and hence one can use the results of Section 3.2.2.2.

Substituting $m^2 = \alpha H^2 = \frac{\alpha}{3}V$ in (3.57), we obtain

$$\sigma^2(\phi) = \frac{1}{12\pi^2}\int_{\phi}^{\phi_i} \frac{V^2(\tilde{\phi})}{V'} \exp\left(-\frac{2\alpha}{3}\int_{\phi}^{\tilde{\phi}} \frac{V}{V'} d\bar{\phi}\right) d\tilde{\phi}. \tag{3.70}$$

For the case of the power-law potential $V$ the integral in (3.70) can be calculated exactly. In particular, for $V = \frac{1}{2}m^2\phi^2$ and $\alpha \gg \phi_i^{-2} \sim m \gg \phi^2$ one obtains

$$\sigma^2(\phi) = \frac{m^2}{16\pi^2\alpha}\left(\phi^2 + \frac{6}{\alpha}\right). \tag{3.71}$$

Note that this result does not depend on the initial value of the inflaton field. At the end of the first stage of inflation when $\alpha\phi_1^2/6 = 1$, both terms in the brackets are equal to each other and the averaged value of $\sigma$ at that time is about

$$\sigma(\phi_1) \simeq \frac{\sqrt{3}\,m}{2\pi\alpha}, \tag{3.72}$$

while the amplitude of the perturbations of field $\sigma$ is

$$\delta\sigma(\phi_1) \simeq \frac{m}{2\pi\sqrt{\alpha}}. \tag{3.73}$$

The CMB normalization of the amplitude of the perturbations thus requires

$$r\frac{\delta\sigma}{\sigma} \simeq r\sqrt{\frac{\alpha}{3}} \sim 7\times 10^{-5}. \tag{3.74}$$

Note that in this case the amplitude of the curvaton perturbations does not depend on the inflaton mass $m$.

Taking $\alpha = 10^{-4}$ we find that $r \approx 10^{-2}$ and hence $f_{\rm NL} \sim 10^2$. Meanwhile for

$\alpha = 10^{-2}$ we should have $r \approx 10^{-3}$, which gives $f_{\rm NL} \sim 10^3$.

One may wonder what is the origin of such an incredible sensitivity of the results to the choice of the parameter $\alpha$. The answer is that this parameter makes the mass of the curvaton field much greater than the mass of the inflaton field at the early stages of inflation. As a result, the distribution of the field $\sigma$ shrinks fast while the field $\phi$ rolls down.

In the calculations above we have ignored the supergravity correction to the curvaton mass squared: $\frac{3}{4}m^2\phi^2\sigma^2 = \frac{9}{2}H^2\sigma^2$. As we will show in the next section this correction can be ignored only if $\alpha \gg 10^{-1}m^{2/3}$ and hence the results of this section are applicable only in this case.

#### 3.2.2.6 $m_\sigma^2 = m^2 + \alpha H^2 + \frac{9}{2}H^2\sigma^2$, $\alpha > 0$

Now we will study the curvaton perturbations in the theory with the general potential

$$V = \frac{m^2\phi^2}{2} + \frac{m^2\sigma^2}{2} + \frac{m^2\phi^2\sigma^4}{16} + \alpha\frac{m^2\phi^2\sigma^2}{6}, \tag{3.75}$$

which corresponds to the curvaton mass (3.48). For $\sigma^2 \ll 1$ Equation (3.56) becomes

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\alpha y}{3} + \frac{y^2}{4} - bx, \tag{3.76}$$

where $x = \phi^2$, $y = \sigma^2$, $b = \frac{m^2}{96\pi^2}$. In this case, unlike to Equation (3.64), there is no exact analytical solution. Nevertheless one can investigate the solutions of this equation using the phase diagram method. If inflation lasts long enough, all solutions, independently of the initial conditions, converge at a certain attractor trajectory in the phase space $(y, x)$, or, equivalently in the space $(\sigma, \phi)$, see Fig. 3.1. For large $\phi$, this attractor trajectory is given in the leading order by the solution of the algebraic equation:

$$\frac{\tilde{y}^2}{4} + \tilde{y}\left(\frac{1}{x} + \frac{\alpha}{3}\right) - bx = 0, \tag{3.77}$$

which is

$$\tilde{y}(x) = -2\left(\frac{1}{x} + \frac{\alpha}{3}\right) + 2\sqrt{\left(\frac{1}{x} + \frac{\alpha}{3}\right)^2 + bx}. \tag{3.78}$$

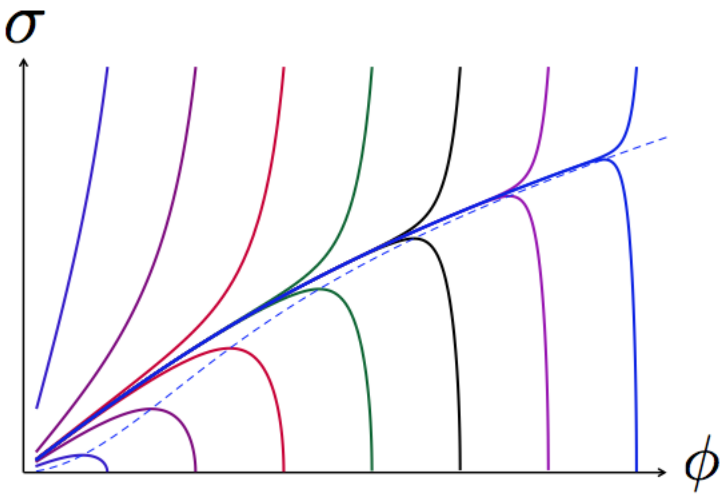


Figure 3.1: Behavior of the average value of the curvaton field $\sigma$ as a function of the inflaton field $\phi$, for various initial conditions. As we see, all trajectories which start at the early stages of inflation (large field $\phi$) converge to the same attractor solution. We follow it until the field $\phi$ becomes $\mathcal{O}(1)$ and inflation ends. At large $\phi$, this solution is very close to the square root of the function (3.78), which is shown by the blue dashed line.

The existence of the attractor solution implies that if inflation is long enough, the final results do not depend on the choice of initial conditions for the curvaton field. We have also found above that in the limit $\alpha \to 0$ one should get the asymptotic solution (3.67), whereas for large $\alpha$ the asymptotic solution is given by (3.71), (3.72). One may wonder how large should $\alpha$ become for the switch between these asymptotic regimes?

To answer this question, let us use the variables:

$$x = z\, b^{-1/3}\,, \qquad y = u\, b^{1/3}\,, \qquad \alpha = \gamma\, b^{1/3}, \tag{3.79}$$

in terms of which Equation (3.76) becomes

$$\frac{du}{dz} = \frac{u}{z} + \frac{\gamma u}{3} + \frac{u^2}{4} - z. \tag{3.80}$$

After rewriting Equation (3.76) in this form it becomes clear that the behavior of the solutions is controlled by a single parameter $\gamma = \alpha\, b^{-1/3} \sim 10\, \alpha\, m^{-2/3}$. One can easily understand that the two asymptotic regimes discussed above corresponds to $\gamma \ll 1$ and $\gamma \gg 1$. One can confirm this conclusion by direct numerical calculations.

This means that the results obtained in Section 3.2.2.4 are valid for $\alpha \ll 10^{-1}m^{2/3}$. Meanwhile in the opposite limit $\alpha \gg 10^{-1}m^{2/3}$ one should use the results of Section 3.2.2.5. To give a particular example, let us take $m \sim 10^{-6}$. In this case one can use the results of Section 3.2.2.4 for $\alpha \ll 10^{-5}$, whereas for $\alpha \gg 10^{-5}$ one should use the results of Section 3.2.2.5.

#### 3.2.2.7 $m_\sigma^2 = m^2 + \alpha H^2 + \frac{9}{2}H^2\sigma^2,\ \ \alpha < 0$

Finally, we will study the case $\alpha < 0$. At first glance in this model the mass squared of the curvaton field at large $H^2$ and $\sigma = 0$ is negative, and therefore one expects a tachyonic instability. However, similar to the model considered in the previous section, one can show that for $|\alpha| \ll 10^{-1}m^{2/3}$ the effect related to the negative mass squared contribution $\alpha H^2$ is subdominant and can be ignored. In this case the results obtained in previous Section 3.2.2.4 are applicable.

For $10^{-1}m^{2/3} \ll |\alpha| \ll 1$, the tachyonic instability leads to spontaneous symmetry breaking controlled by the supergravity correction $\frac{9}{2}H^2\sigma^2 = \frac{3}{4}m^2\phi^2\sigma^2$ to the curvaton mass squared. Indeed, one can show that the minimum of the supergravity potential for the curvaton field, in the regime with $|\alpha|, \sigma \ll 1$, can be found from the following equation:

$$\sigma^2 = \frac{2|\alpha|}{3} + \frac{4}{\phi^2}. \tag{3.81}$$

Therefore at large $\phi$ and $\alpha < 0$ the potential has a minimum at

$$\sigma^2 = \frac{2|\alpha|}{3}. \tag{3.82}$$

This means that during inflation the field $\sigma$ falls towards this minimum, and its distribution becomes centered not at $\sigma = 0$ but at $\sigma = \sqrt{\frac{2|\alpha|}{3}}$. As for the height of the potential along the trajectory with $\sigma = \sqrt{\frac{2|\alpha|}{3}}$, for small $|\alpha|$ it remains approximately given by $m^2\phi^2/2$.

When the field $\phi^2$ becomes smaller than $6/|\alpha|$, the minimum of the potential shifts towards $\sigma = 0$, and the curvaton mass squared becomes equal to $m^2$. However, this does not mean that the distribution of the field $\sigma$ instantly follows the position of the minimum. Since the mass of the field $\sigma$ at that time is much smaller than $H$, the field $\sigma$ will move towards $\sigma = 0$ very slowly, decreasing at the same rate as the amplitude of perturbations $\delta\sigma$. As before, we are assuming that $|\alpha| \lesssim 1/40$, and therefore the curvaton mass squared is given by $m^2$ during the last 60 e-folds of inflation. This leads to the following result for the perturbations:

$$\frac{\delta\sigma}{\sigma} \sim \frac{m\sqrt{3}}{2\sqrt{2}\pi\,|\alpha|}. \tag{3.83}$$

For $m \sim 10^{-7}$ and $\alpha \sim -10^{-4}$ a proper amplitude of perturbations corresponds to $r \approx 1/3$ and, hence, $f_{\rm NL} = \mathcal{O}(3)$. However, one can easily increase $f_{\rm NL}$ by increasing $m$ and/or decreasing $|\alpha|$. For example, taking $m \sim 10^{-7}$ and $\alpha \sim -10^{-5}$ gives $f_{\rm NL} \sim 30$.

#### 3.2.2.8 Non-Gaussianity and the curvaton web

In the previous sections we have evaluated the average value of the curvaton field at the last stages of inflation, and calculated the parameter $f_{\rm NL}$ describing local non-Gaussianity. However, we should remember that when we were talking about the classical homogeneous curvaton field $\sigma$, we had in mind the long-wavelength perturbations which look homogeneous on the scales corresponding to the present observable part of the Universe. In reality this classical field in our model is a random variable with the expectation value $\bar{\sigma} = \sqrt{\langle\delta\sigma^2\rangle}$ obtained by summing up the contributions of all long-wavelength fluctuations (larger than the present horizon) generated on inflation. All calculations above were performed taking $\sigma$ to be equal $\bar{\sigma}$. However, because $\sigma$ is a random Gaussian variable it takes different values in different parts of the Universe of the size of our horizon [80].

To evaluate the observational implications of this fact, let us try to understand how the amplitude of perturbations of the metric and the local value of $f_{\rm NL}$ depend on the local value of $\sigma$. For simplicity, we will assume that the standard inflaton perturbations are very small, so that we can ignore them in our investigation. This can be achieved by considering a model with $m \ll 6 \times 10^{-6}$. We will also assume that the curvaton field density at the moment of the curvaton decay is much smaller than the total density, i.e. $r \ll 1$. In this case, the change of $\sigma$ does not affect the total density $\rho$, but it does affect $\delta\rho(\sigma)$, which is proportional to $\sigma$. This means that the amplitude of perturbations of the metric produced by fluctuations of the curvaton field will be proportional to $\frac{\sigma}{\bar{\sigma}}$:

$$\frac{\delta\rho(\sigma)}{\rho} = \frac{\delta\rho(\bar{\sigma})}{\rho} \cdot \frac{\sigma}{\bar{\sigma}}. \tag{3.84}$$

Meanwhile the local value of $f_{\rm NL}$ is inversely proportional to $r = \frac{\rho(\sigma)}{\rho}$. For small $\sigma$, the value of $\rho(\sigma)$ is proportional to $\sigma^2$. Therefore

$$f_{\rm NL}(\sigma) = f_{\rm NL}(\bar{\sigma}) \cdot \frac{\bar{\sigma}^2}{\sigma^2}. \tag{3.85}$$

The probability that the curvaton field will take some value much greater than $\bar{\sigma}$, is exponentially small. However, the probability that $\sigma$ is substantially smaller than $\bar{\sigma}$ can be rather large. To estimate this probability we will make a simplifying assumption. Namely, we assume that all values of the field $S = \sigma\, e^{i\theta}/\sqrt{2}$ with $|S| < \bar{\sigma}$ are equally probable, but the probability vanishes for $|S| > \bar{\sigma}$. The maximal value of the curvaton field is $\bar{\sigma}\sqrt{2}$, the probability to find the field $\sigma$ in the interval $d\sigma$ from 0 to $\bar{\sigma}\sqrt{2}$ is given by $\frac{\sigma d\sigma}{\bar{\sigma}^2}$, and the average value of the curvaton field is $\bar{\sigma}$, as it should be.

Now let us evaluate the average value of the amplitude of density perturbations, averaged over all possible values of $\sigma$:

$$\left\langle \frac{\delta\rho(\sigma)}{\rho} \right\rangle \simeq \frac{\delta\rho(\bar{\sigma})}{\rho} \int\limits_0^{\bar{\sigma}\sqrt{2}} \frac{\sigma}{\bar{\sigma}} \frac{\sigma d\sigma}{\bar{\sigma}^2} \simeq \frac{2\sqrt{2}}{3} \frac{\delta\rho(\bar{\sigma})}{\rho}. \tag{3.86}$$

Thus, the average amplitude of the curvaton perturbations almost exactly coincides with the amplitude of perturbations in the Universe with an average curvaton field $\bar{\sigma}$.

However, the situation with $\langle f_{\rm NL} \rangle$ is quite different. Since $f_{\rm NL}$ is proportional to $\sigma^{-2}$, its expectation value over the whole Universe acquires a divergent contribution from the parts of the Universe with small $\sigma$. Our calculations are valid only for fluctuations produced well before the last 60 e-folds of inflation, with a combined amplitude $\sigma$ above $\mathcal{O}(H)$. Introducing the cut-off at $\sigma \sim H \sim 2\pi\delta\sigma$, we find

$$\langle f_{\rm NL}\rangle \simeq f_{\rm NL}\left(\bar{\sigma}\right)\int\limits_{H}^{\bar{\sigma}\sqrt{2}}\left(\frac{\bar{\sigma}}{\sigma}\right)^2\frac{\sigma d\sigma}{\bar{\sigma}^2} \simeq f_{\rm NL}\left(\bar{\sigma}\right)\ln\left(\frac{\bar{\sigma}}{\sqrt{2}\,\pi\delta\sigma}\right). \tag{3.87}$$

How significant is the effect discussed above? To give a particular numerical example, let us consider the case $\alpha = 0$. In this case, according to Equation (3.68), one has $\frac{\delta\sigma}{\bar{\sigma}} \sim 0.4\, m^{1/3}$. We found that for $m \sim 10^{-7}$ one has $f_{\rm NL}(\bar{\sigma}) \sim 30$. In this case Equation (3.87) implies that $\langle f_{\rm NL}\rangle \sim 5\, f_{\rm NL}(\bar{\sigma}) \sim 150$.

Thus we deal with a significant effect of statistical amplification of non-Gaussianity: Even though the fraction of the volume of the Universe with $f_{\rm NL}(\sigma) \gg f_{\rm NL}(\bar{\sigma})$ is relatively small, the values of $f_{\rm NL}$ in those parts of the Universe can be huge, so the expectation value of $f_{\rm NL}$ can be much greater than the value of this parameter $f_{\rm NL}(\bar{\sigma})$ calculated in the previous sections.

This effect becomes even stronger in the models where the curvaton field is real (instead of being a radial part of a complex field). In such models

$$\langle f_{\rm NL}\rangle \simeq f_{\rm NL}\left(\bar{\sigma}\right)\int\limits_{H}^{2\bar{\sigma}}\left(\frac{\bar{\sigma}}{\sigma}\right)^2\frac{d\sigma}{\bar{\sigma}} \simeq f_{\rm NL}\left(\bar{\sigma}\right)\frac{\bar{\sigma}}{2\,\pi\delta\sigma}. \tag{3.88}$$

In the particular example discussed above, $\frac{\delta\sigma}{\bar{\sigma}} \sim 0.4\, m^{1/3}$ and $m \sim 10^{-7}$, this would lead to an enormously large amplification effect: $\langle f_{\rm NL}\rangle \sim 10^2\, f_{\rm NL}(\bar{\sigma}) \sim 3000$.

Thus we see that in the curvaton scenario some fraction of the Universe can be in a state with the curvaton field $\sigma$ significantly smaller than its average value $\bar{\sigma}$. In such parts of the Universe, the locally observed level of non-Gaussianity will strongly exceed its value $f_{\rm NL}(\bar{\sigma})$ calculated in the previous sections. This effect is so significant that the average value of the parameter $f_{\rm NL}$ can be much greater than the value $f_{\rm NL}$ in the part of the Universes with an average value of the field $\sigma$. In other words, operations of averaging in this case are not commutative.

For a complete investigation of this effect one should also take into account the standard inflationary perturbations of metric. The curvaton perturbations are important only in the cases where the standard inflaton perturbations are suppressed. That is why we assumed that $m \ll 6 \times 10^{-6}$. But the standard inflaton perturbations may dominate in the rare parts of the Universe where $\sigma \ll \bar{\sigma}$. In such cases one should perform a more detailed investigation of non-Gaussianity of perturbations produced by all sources.

This means that one should be very careful when formulating predictions for the non-Gaussianity parameter $f_{\mathrm{NL}}$ in the curvaton scenario, because the distribution of possible values of $f_{\mathrm{NL}}$ in the curvaton web can be extremely broad. Moreover, the existence of the anti-correlation between the amplitude of the perturbations of the metric $\left(\frac{\delta\rho(\sigma)}{\rho}\right)^2$ and the non-Gaussianity parameter $f_{\mathrm{NL}}$ for $r \ll 1$ (see Equations (3.84) and (3.85)) suggests that anthropic considerations may play a very important role in the evaluation of the probability to live and make observations in parts of the curvaton web with different values of the non-Gaussianity parameter $f_{\mathrm{NL}}$ [46, 47, 80–82, 117].

The difference between $\langle f_{\mathrm{NL}} \rangle$ and $f_{\mathrm{NL}}(\bar{\sigma})$ clearly demonstrates that $f_{\mathrm{NL}}$ is not a perfect tool for the description of non-Gaussianity. As shown in [80], the distribution of the regions of small (large) perturbations of metric and spikes of non-Gaussianity has an interesting structure, which we called "the curvaton web". This structure has a non-perturbative origin.

Indeed, the non-Gaussianity parameter $f_{\mathrm{NL}}(\sigma)$ takes its largest values in the regions of the Universe where the classical curvaton field $\sigma$ is small, see (3.85). In the theories where the curvaton field is a real, single component field, the regions of small $\sigma$ correspond to domain walls separating large domains with $\sigma > 0$ from large domains with $\sigma < 0$ [80].

In the theory studied in the present work, the curvaton field $\sigma$ corresponds to the radial component of a complex field $S$. In this case, the regions of small $\sigma$ form strings, reminiscent of the cosmic strings which appear due to spontaneous symmetry breaking. In our case, however, unlike in the usual cosmic string case, the curvaton strings appear in the places corresponding to the *minimum* of energy of the curvaton field. If one considers more complicated models, where the curvaton has $O(3)$ symmetry, instead of the domain walls and cosmic strings one will have localized objects reminiscent of global monopoles. In other words, the distribution of the peaks of non-Gaussianity in the curvaton scenario has topological origin, which cannot be fully described by the standard tools of perturbation theory, such as $f_{\rm NL}$ and $g_{\rm NL}$.

#### 3.2.2.9 Discussion

We discussed the curvaton scenario, which naturally emerges in the simplest supergravity realization of the chaotic inflation scenario [63, 64, 66]. The investigation of this scenario consists of several parts. The main step is to find an average value of the curvaton field $\sigma$ after a long stage of inflation. One needs this to calculate the amplitude of perturbations of the density of the curvaton field. We performed this investigation by analyzing the growth of the curvaton perturbations during inflation.

To conclude this investigation, one should find the ratio $r$ of the energy of the curvaton field to the energy density of all other particles and fields at the time of the curvaton decay. This is a complicated and model-dependent problem, which requires the study of reheating after inflation, the decay rate of the curvaton field, and the composition of matter at the time of the curvaton decay. Here we simply treated $r$ as a free phenomenological parameter, but one should remember that all of the issues mentioned above should be addressed in a more detailed investigation.

We analyzed the model with the simplest quadratic inflaton potential and with the curvaton mass given by $\alpha H^2 + m^2$. Our investigation demonstrates that if inflation is long enough, then the average value of the curvaton contribution to the amplitude of metric perturbations, as well as the averaged value of the non-Gaussianity parameter $f_{\rm NL}$, do not depend on initial conditions for the curvaton field. The final results depend on the inflaton mass $m$, and on the parameter $\alpha$, which is related to the curvature of the Kähler manifold [64]. However, the locally observable parameter $f_{\rm NL}$ and the amplitude of the curvaton perturbations may take different values in different parts of the Universe and in certain cases they may significantly deviate from their averaged values [80]. Moreover, the average value of the parameter $f_{\rm NL}$ can be much greater than the value $f_{\rm NL}$ in the part of the Universe with an average value of the field $\sigma$. For a certain choice of parameters, the value of the non-Gaussianity parameter $f_{\rm NL}$ can be in the observationally interesting range from $\mathcal{O}(10)$ to $\mathcal{O}(100)$.

The curvaton perturbations in our simple model have a flat spectrum. This is a consequence of the degeneracy of the masses of the inflaton and curvaton field at the end of inflation. One can change the spectral index by switching to a theory with a different inflaton potential. This can be easily realized in the new class of chaotic inflation models in supergravity, or by splitting the spectrum of fluctuations of the curvaton field into two branches with different masses [63, 64]. The last possibility can be realized by modifying the Kähler potential, or by adding a term $\sim S^3$ to the superpotential.

Another interesting possibility is to take the inflaton mass just a little bit smaller than $m \sim 6 \times 10^{-6}$, to decrease the amplitude of the standard inflaton perturbations. Then one may compensate for this decrease by adding a small contribution of the curvaton fluctuations. This will result in a smaller amplitude of tensor modes and a larger spectral index $n_s$, which would improve the agreement of the predictions of the simplest chaotic inflation models with the WMAP data. Also, as our calculations demonstrate, for certain values of parameters even a small contribution of the curvaton perturbations may dramatically increase the non-Gaussianity of the combined spectrum of perturbations of metric.

Thus, whereas the curvaton models are more complicated than the single-field inflationary models, they make the resulting scenario much more flexible, which may be important for a proper interpretation [38] of the coming observational data.

Our final comment deals with the topological features of the distribution of perturbations in the curvaton scenario. We point out that in the theory of a single-component real curvaton field, the regions of the Universe with large non-Gaussianity form domain walls [80], reminiscent of the exponentially thick cosmic domain walls. Meanwhile in the theory of a complex curvaton field, which was studied here, the regions of large non-Gaussianity form exponentially thick cosmic strings. In more complicated theories, these regions may form separate islands of large local non-Gaussianity, resembling global monopoles. Since these effects have a non-perturbative, topological origin, non-Gaussianity in the curvaton scenario cannot be fully described by such tools as the familiar perturbation theory parameters $f_{\rm NL}$ and $g_{\rm NL}$.

# Chapter 4

# Magnetic fields in the early Universe

Astronomical observations show that all celestial bodies carry magnetic fields. From planets to interstellar medium, fields of varying strength and extension have been measured. A particularly interesting case are galaxies, galaxy clusters and beyond, the intergalactic medium and the Universe at large. These fields are of the order of a few micro Gauss and they extend over kiloparsecs or more. Unfortunately their structure is not always simple. Beside a constant component they have a complex structure with varying symmetry, which shows that processing has taken place since their appearance.

In this chapter we want to review the effects and the production mechanisms of magnetic fields. In Section 4.1 we describe the observational methods used to measure galactic and extra-galactic magnetic fields and we summarize the observations of the typical strength detected in galaxies, clusters of galaxies and the limits on the intergalactic magnetic fields. In Section 4.4 we describe the effects of a magnetic field on the CMB and in Section 4.5 we consider the constraints on the amplitude of magnetic fields set by Big Bang nucleosynthesis. Finally in Section 4.6 we review several mechanisms for magnetogenesis. In this chapter we follow [51, 65, 126].

## 4.1 Large-scale magnetic fields in the Universe

## 4.2 Observational methods

There are mainly three methods for the observation of galactic and extra-galactic magnetic fields: the intensity and the polarization of synchrotron emission from free relativistic electrons, the Faraday rotation measurements of polarized radiation passing through an ionized plasma and the Zeeman splitting of spectral lines. The Zeeman splitting is a direct method for detecting the magnetic field, but it is very difficult to observe and at present there are no confirmed detections in systems beyond the galaxy. Synchrotron emission and Faraday rotation allow to measure magnetic fields in very distant objects.

### 4.2.1 Synchrotron emission

Synchrotron radiation refers to the radiation produced when relativistic electrons interact with a magnetic field. It is used to study magnetic fields ranging from pulsars to superclusters. The total synchrotron emission provides the strength of the magnetic field and the degree of polarization gives informations about the field's uniformity and structure.

### 4.2.2 Faraday rotation

The magnetic field in the intracluster medium in clusters of galaxies can be measured direclty through the effect of the field on the propagation of the linearly polarized radiation. When passing through an ionized and magnetized plasma, linearly polarized radiation experiences Faraday rotation, namely a rotation in time of the electric field vector.

### 4.2.3 Zeeman splitting

In vacuum, an atom has several electronic configurations with the same energy, the electronic energy levels are independent of the direction of the angular momentum vector. A magnetic field breaks this degeneracy by splitting neighboring energy levels. This is the most direct method for observing magnetic fields, but it is very difficult to detect outside our galaxy.

## 4.3 Observations of magnetic fields

Magnetic fields in galaxies are determined using several methods. Typical strength in spiral galaxies are around 10 $\mu$G. Radio faint galaxies have weaker fields, 5 $\mu$G, while starburst and merging galaxies have the strongest field, between 50 and 100 $\mu$G. The magnetic structure observed can be symmetrical and in some cases there is no recognizable field structure [22].

Magnetic fields have also been measured in clusters of galaxies, through synchrotron emission of electrons spiralling along the field lines and Faraday rotation measurements of polarized emissions crossing the intracluster medium. The strength detected within clusters of galaxies is of 1 - 10 $\mu$G and varies slightly with the type of cluster. The field structure is not homogeneous, indicating the presence of a tangled magnetic field [124].

High resolution Faraday rotation measurements have detected magnetic fields in high redshift objects, such as very far quasars. A particular example is the quasar 3C191 at $z = 1.945$ with a field strength in the range of 0.4 - 4 $\mu$G [43].

The intergalactic medium is also permeated with magnetic fields. Recently limits on intergalactic fields have been found using combined data from Atmospheric Cherenkov Telescopes and Fermi Gamma-Ray Space Telescope based on the spectra of three blazars, $1 \times 10^{-17}$ G $< B < 3 \times 10^{-14}$ G [44]. Using data from HESS and Fermi a lower bound of $\sim 10^{-15}$ G was imposed [31, 100, 116]. Measurements of intergalactic magnetic fields are very important because they may allow to distinguish between a cosmological and an astrophysical origin of the fields and this may open a new window in the understanding of the physics of the Early Universe.

## 4.4 Effects on the Cosmic Microwave Background

### 4.4.1 A constant magnetic field

A spatially constant magnetic field affects the geometry of the Universe introducing a shear. The electromagnetic energy-momentum tensor acts as a source in the Einstein equations and has the form [51]

$$T^{\alpha\beta}_{em} = \frac{1}{4\pi}\left(-F^{\alpha\mu}F^{\beta}_{\mu} + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right), \tag{4.1}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. In the presence of a homogeneous magnetic field directed along the $z$ axis

$$T^{00} = T^{11} = T^{22} = -T^{33} = \rho_B = \frac{B^2}{8\pi}, \qquad T^{0i} = 0, \tag{4.2}$$

the energy-momentum tensor becomes anisotropic with a positive pressure along the $x$ and $y$ axes and a negative pressure along the $z$ axis. This anisotropic pressure gives rise to an anisotropic expansion law. Let us consider the most axially symmetric model with metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2) - b^2(t)dz^2, \tag{4.3}$$

and define $\alpha = \dot{a}/a$, $\beta = \dot{b}/b$ and $r \equiv \rho_B/\rho_{rad}$, $\sigma \equiv \alpha - \beta$. Assuming $r < 1$ and $\sigma < 1$ the Einstein equations will take the following form

$$\frac{d}{dt}\left(\frac{\sigma}{H}\right) = -\left(\frac{\sigma}{H}\right)\frac{\gamma - 2}{\gamma t} + \frac{4r}{\gamma t}, \tag{4.4}$$

$$\frac{dr}{dt} = -\frac{2r}{9\gamma t}\left(4\frac{\sigma}{H} + 9\gamma - 12\right), \tag{4.5}$$

where $H = (2\alpha + \beta)$ and $p = (1 - \gamma)\rho$. Substituting the asymptotic value $\sigma \to 6r$ into (4.5) during the radiation era one finds

$$r(t) = \frac{q}{1 + 4q\ln(t/t_0)}, \tag{4.6}$$

where $q$ is a constant. Thus the ratio of the magnetic and blackbody radiation densities is changing logarithmically during the radiation era. We assume that at the recombination time $t_{rec}$ the temperature is everywhere $T_{rec}$. At the present time $t_0$ the temperature of relic photons coming from the three directions is

$$T_{x,y} = T_{rec}\frac{a}{a_0} = T_{rec}\exp\left(-\int_{t_{rec}}^{t_0} \alpha dt\right), \tag{4.7}$$

$$T_z = T_{rec}\frac{b}{b_0} = T_{rec}\exp\left(-\int_{t_{rec}}^{t_0} \beta dt\right). \tag{4.8}$$

Therefore the temperature anisotropy is

$$\frac{\Delta T}{T} = \frac{T_x - T_z}{T_{rec}} \approx -\frac{1}{2}\int_{t_{rec}}^{t_0} \sigma d\ln t. \tag{4.9}$$

This means that a magnetic field, which today has a strength of $10^{-9} - 10^{-10}$ G, would produce a temperature anisotropy $\Delta T/T \lesssim 10^{-6}$.

Barrow, Ferreira and Silk [11] derived an upper limit on the present strength of any primordial homogeneous magnetic field calculating the microwave background anisotropy created by cosmological magnetic fields. They considered the cosmological evolution of the most general homogeneous magnetic fields, calculated their gravitational effect on the temperature anisotropy of the microwave background radiation and derived a limit on the strength of the field using the 4-year COBE microwave background measurements. They obtained the limit

$$B(t_0) < 3.5 \times 10^{-9} f^{1/2}(\Omega_0 h_{50}^2)^{1/2}\mathrm{G}, \tag{4.10}$$

where $f$ is a $O(1)$ shape factor which accounts for possible non-Gaussian characteristics of the COBE data. From this result we see that COBE data are not incompatible with magnetic fields of primordial origin.

### 4.4.2 The effect on the acoustic peaks

The presence of a sizable magnetic field has an effect on the acoustic peaks of the CMB. Before the last scattering when primordial density fluctuations, generated on inflation, enter the horizon they create acoustic oscillations in the plasma.

These oscillations distort the primordial spectrum of anisotropies. The presence of a primordial magnetic field will also affect the plasma oscillations.

We consider a magnetic field which is homogeneous on scales larger than the scale of plasma oscillations. We assume that the medium has an infinite electric conductivity, such that the magnetic flux is constant in time, and that there is no dissipative effect, $\lambda = 2\pi/k \gg l_{diss}$. The magnetic field is $\boldsymbol{B}_0 + \boldsymbol{B}_1$, where $\boldsymbol{B}_0$ is the background field, which is constant in space, and $\boldsymbol{B}_1$ is a small perturbation. Since the Universe is expanding, $\boldsymbol{B}_0 \propto a^{-2}$. In this framework the equations of magnetohydrodynamics in comoving coordinates are

$$\dot{\delta} + \frac{\nabla \cdot \boldsymbol{v}_1}{a} = 0, \tag{4.11}$$

$$\dot{\boldsymbol{v}}_1 + \frac{\dot{a}}{a}\boldsymbol{v}_1 + \frac{c_s^2}{a}\nabla\delta + \frac{\nabla\phi_1}{a} + \frac{\hat{\boldsymbol{B}}_0 \times (\dot{\boldsymbol{v}}_1 \times \hat{\boldsymbol{B}}_0)}{4\pi a^4} + \frac{\hat{\boldsymbol{B}}_0 \times (\nabla \times \hat{\boldsymbol{B}}_1)}{4\pi\rho_0 a^5} = 0, \tag{4.12}$$

$$\partial_t \hat{\boldsymbol{B}}_1 = \frac{\nabla \times (\boldsymbol{v}_1 \times \hat{\boldsymbol{B}}_0)}{a}, \tag{4.13}$$

$$\nabla^2\phi_1 = 4\pi G\rho_0 \left(\delta + \frac{\hat{\boldsymbol{B}}_0 \cdot \hat{\boldsymbol{B}}_1}{4\pi\rho_0 a^4}\right), \tag{4.14}$$

$$\nabla \cdot \hat{\boldsymbol{B}}_1 = 0, \tag{4.15}$$

where $\hat{\boldsymbol{B}} \equiv \boldsymbol{B}a^2$, $\delta = \rho_1/\rho_0$, $v_1$ are small perturbation on the background density, gravitational potential and velocity respectively, and $c_s$ is the sound velocity.

If there is no magnetic field, there is the ordinary sound wave involving density fluctuations and longitudinal velocity fluctuations. If a magnetic field is present, there are three different waves. Two types of scalar waves, the fast and slow magneto-sonic waves, and the Alfvén waves. Fast magneto-sonic waves are ordinary sound waves which are modified by the presence of the magnetic field, their velocity is given by $c_+^2 \sim c_s^2 + (\boldsymbol{k} \cdot \boldsymbol{B})^2/(4\pi\rho)$. Slow magneto-sonic waves determine a new form of wave due to the interaction of the charged plasma with the magnetic field. The third type of waves are the Alfvén waves, they are vector perturbations in the plasma velocity, which oscillates. They are generated because of the coupling of the magnetic field to the charged electron-proton plasma. Fast magneto-sonic waves cause a slight shift of the acoustic peaks, see Figure 4.1. This shift might be detectable in CMB experiments.

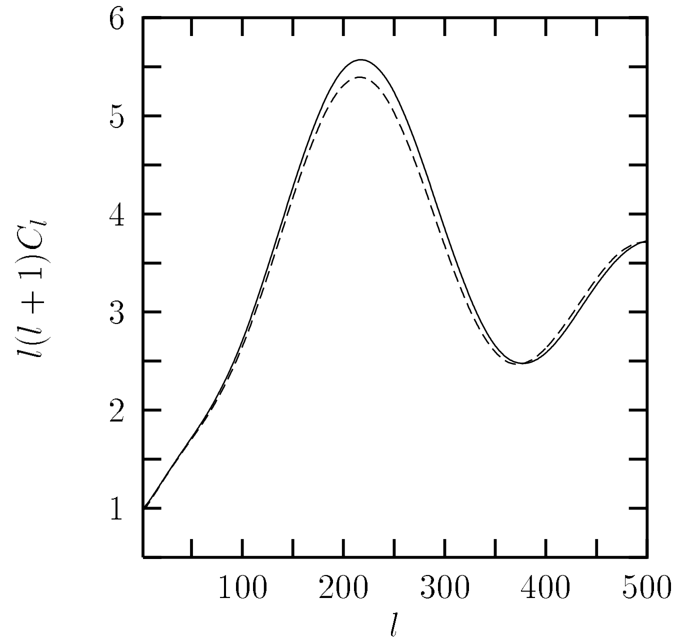


Figure 4.1: Modification of the CMB anisotropy spectrum due to the presence of a magnetic field of strength $2 \times 10^{-7}$ Gauss (dashed line). The solid line shows the prediction of standard CDM cosmology. From [1].

## 4.5 Constraints from Big Bang Nucleosynthesis

Big Bang Nucleosyntesis (BBN) took place between $10^{-2}$ and 1 s after the Big Bang and is responsible for most of the $^4$He, $^3$He, D and $^7$Li in the Universe. The observational data can be compared to detailed predictions of numerical calculations. Magnetic fields can alter the predictions of BBN, thus BBN implies limits on the strength of primordial magnetic fields.

The main effects of the presence of a magnetic field on BBN are related with: the proton-to-neutron conversion ratio, the expansion and cooling of the Universe and the electron thermodynamics. Let us briefly explain how a magnetic field affects them. Firstly, in the early Universe the weak interaction is responsible for maintaining chemical equilibrium between protons and neutrons. A strong magnetic field during nucleosynthesis would enhance the conversion rate of neutrons into protons and therefore the neutron-to-proton ratio would freeze out at lower temperature. The result would be a less efficient production of $^4$He and of heavier elements [90]. The effect would be too big if $B \gg M_P^2/e \sim 10^{17}$ G at the time of nucleosynthesis. Secondly, the balance between the time-scale of the weak interaction and the expansion rate of the Universe determines the temperature at which the proton-to-neutron ratio freezes out. There is equilibrium when $\Gamma_{n\to p} \sim H$, where $\Gamma_{n\to p}$ is the cross-section of the interaction and $H$ is the Hubble parameter at that time. $H$ is proportional to the total energy density of the Universe where the magnetic field is present. Therefore if the magnetic field is strong, the value of the Hubble parameter would increase and it would cause an earlier freeze-out of the proton-to-neutron ratio and result into a larger residual amount of $^4$He [50, 67]. Finally, a magnetic field would also change the phase space volume of electrons and positrons, because their momentum component normal to the magnetic field would become discrete. So there would be an increase of the energy density, the number density and the pressure of the electron gas, with respect to the case without a magnetic field. The increase would make the photons transfer energy to the lowest Landau level and this would delay the electron-positron annihilation, which in turn increases the photon-to-baryon ratio and leads to a lower $^3$He and D abundances [72].

Numerical calculations which take into account all these effects conclude that the main consequence on the light elements abundance is given by the field's contribution to the expansion rate of the Universe. The overall constraint on the magnetic field amplitude is $B \lesssim 7 \times 10^{-7}$ G at the time of galaxy formation [51].

## 4.6 Generation of large-scale magnetic fields

Many astrophysicists believe that large-scale magnetic fields are generated and maintained by a dynamo mechanism, which is responsible for the conversion of kinetic energy of an electrically conducting fluid into magnetic field energy. A seed field is needed to initiate the dynamo process and various mechanisms have been proposed to generate this seed field. An alternative to the dynamo mechanism is to consider that the magnetic field measured today is the primordial one and that it was not amplified by the dynamo mechanism. In the following we sketch the idea of the dynamo mechanism and we briefly describe some of the methods proposed in the literature to directly generate a magnetic field or to obtain a seed field for the dynamo mechanism.

### 4.6.1 The dynamo mechanism

The most common approach to the dynamo mechanism is the mean field dynamo [51]. It is based on the assumption that fluctuations in the magnetic and velocity fields are much smaller than the mean slowly varying components of the corresponding quantities. By a suitable averaging of the equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{4\pi\sigma}\nabla^2 \boldsymbol{B}, \tag{4.16}$$

where $\sigma$ is the electric conductivity, one obtains the temporal evolution of the mean component of the magnetic field

$$\frac{\partial \boldsymbol{B}_0}{\partial t} = \nabla \times (\alpha \boldsymbol{B_0} + \boldsymbol{v_0} \times \boldsymbol{B_0}) - \nabla \times [(\eta + \beta)\nabla \times \boldsymbol{B_0}], \tag{4.17}$$

where

$$\alpha = -\frac{1}{3}\tau_c \langle \boldsymbol{v}_1 \cdot \nabla \times \boldsymbol{v}_1 \rangle, \qquad \beta = \frac{1}{3}\tau_c \langle \boldsymbol{v}_1^2 \rangle, \tag{4.18}$$

$\eta = 1/4\pi\sigma$ is the magnetic diffusivity and $\tau_c$ is the correlation time for the ensemble of random velocities. The coefficient $\alpha$ is proportional to the helicity of the flow $h = \langle \boldsymbol{v}_1 \cdot \nabla \times \boldsymbol{v}_1 \rangle$, which measures the level to which the streamlines are twisted. In order to have $\alpha \propto h \neq 0$ a macroscopic parity violation is required. A possible source of this violation can be the Coriolis force produced by the rotation of the galaxy. If the $\beta$ term can be neglected, the solution of (4.17) is

$$\boldsymbol{B}_0 = (\pm \sin kz, \cos kz, 0)e^{\gamma t}, \tag{4.19}$$

where $z$ is the coordinate along the galaxy rotation axis and $\gamma = -\eta k^2 \pm \alpha k$, $k \sim 1/L$ is the wavenumber. The field grows exponentially with time if the helicity $h$ is non-zero and if the scale $L$ is sufficiently large. Amplification ends when there is equipartition between the kinetic energy density of the small-scale turbulent fluid motion and the magnetic energy density. Depending on the details of the model and on the properties of the medium, in the case of a Universe dominated by CDM with no cosmological constant, the time to reach saturation starting from a seed field of strength $10^{-20}$ G might be $10^8 - 10^9$ years. In the case of a Universe with a cosmological constant, the required seed field might be

$10^{-30}$ G [24].

### 4.6.2 Primordial vorticity

Harrison [54] proposed that primordial magnetic fields can be generated during the radiation era by plasma vortical motion. The idea is that Thomson scattering is more efficient for electrons than for ions so their rotational velocities decrease differently in the expanding Universe in the pre-recombination era. The magnetic field is generated by an electric current due to an electromotive force created by the difference between the angular velocities of electrons and ions. Indeed during the expansion the angular velocity of electrons decreases as $\omega \propto a^{-1}$ and the one of ions as $\omega \propto a^{-2}$, where $a$ is the scale factor. In [55] the author shows that it is possible to generate a field that at present time has strength $10^{-8}$ G on a scale of 1 Mpc.

However this scenario is not problem free. Rotational, or vector, density perturbations decay with the Universe expansion, therefore in order to produce sizable effects at recombination time, these perturbations should have been dominant at the radiation-matter equality. This seems in disagreement with the standard scenario for galaxy formation [109]. Moreover a consequence of the Helmholtz-Kelvin circulation theorem, which states that the circulation around a closed curve following the motion of matter is conserved, is that small deviations from the isotropic Friedmann Universe cannot generate rotational perturbations.

In [122] Vilenkin noticed that parity-violating currents may develop in a vortical thermal background as a consequence of the parity violation in the Weinberg-Salam model of the electroweak interaction. In [123] Vilenkin and Leahy proposed that these currents can generate a strong magnetic field.

### 4.6.3 Magnetic fields from the electroweak phase transition

The electroweak phase transition takes place at $T_{ew} \sim 100$ GeV. It is the transition from a symmetric phase with massless gauge bosons to the Higgs phase, in which the $SU(2) \times U(1)_Y$ gauge symmetry is spontaneously broken and all the masses of the model are generated. First order phase transitions take place by bubble nucleation. The new bubbles contain phases of broken symmetry, whose sizes are at most of the order of the horizon at that time. As the Universe expands, different domains come into causal contact and bubble walls collide with each other. The generation of magnetic fields is given by processes that occur during these collisions. Second order phase transitions occur in a regular way and with an approximate thermal equilibrium throughout the process. A magnetic field can be created also in this case.

Following [65], the first proposal for a magnetogenesis mechanism based on first order phase transitions is due to Hogan in [58], where he tried to explain the effects of the fields on structure formation. In this case during the transition free energy is orderly released, inducing a dynamo in the wall of the bubble. Thus each bubble is an independent dynamo producing fields correlated on the scale of the bubble. The result are randomly oriented field lines that properly averaged produce a large field which extends over regions that are not causally connected.

Baym, Bödeker and McLerran [16] proposed a dynamo mechanism where seed fields are provided by thermal fluctuations. The walls of the broken symmetry bubbles expand creating supersonic shock waves. The collision of these shock waves generates a turbulent dynamo that amplifies the field. The amplification proceeds in the following way. The Universe supercools below a critical temperature, $T_{cr} \sim 100$ GeV, then the Higgs field tunnels locally from the unbroken $SU(2) \times U(1)_Y$ phase to the broken $U(1)_{em}$ phase. The tunneling forms broken phase bubbles that expand and convert the false vacuum energy into kinetic energy. As the shock fronts collide, turbulence forms in the cones of the bubble intersection. The magnetic field generated by this mechanism at the present time is $B(l_{gal}) \sim 10^{-17} - 10^{-20}$ G on galactic scales $l_{gal} \sim 10^9$ AU.

In [120] Vachaspati proposed the generation of magnetic fields by second order phase transitions. Below the critical temperature of the electroweak phase transition, $T_{cr}$, the minimum energy state of the Universe corresponds to a spatially homogeneous vacuum in which the Higgs field $\Phi$ is covariantly constant, namely $D_\mu \Phi = (\partial_\mu - ieA_\mu)\Phi = 0$. However, during the phase transition and immediately after it, thermal fluctuations give rise to a finite correlation length, $\xi \sim (eT_{cr})^{-1}$, leading to spatial variations both in the Higgs field module and in its $SU(2)$ and

$U(1)_Y$ phases. The variation of the Higgs field results in the presence of electromagnetic fields. This magnetic field does not need to be the Maxwell magnetic field, it can be the magnetic field associated with any of the unbroken symmetries at the epoch of the phase transition. The value of the field at the present time for a scale of 100 Kpc is approximately $B \sim 10^{-30}$ G.

In [2] and in [39] Ahonen and Enqvist studied the case of an Abelian Higgs model with the formation of ring-like magnetic fields in collisions of bubbles of broken phase. The magnetic field which is formed is of the order $B \sim 10^{20}$ G with a coherence length of about $10^2$ GeV$^{-1}$. They found that when the plasma is endowed with magnetohydrodynamical turbulence, the coherence scale can be enhanced by the inverse cascade of the magnetic helicity and the field can reach the value $B \sim 10^{-21}$ G on a comoving scale of 10 Mpc today.

Grasso and Riotto [49] considered the generation of magnetic fields during a second order electroweak phase transition. They showed that the field generation is intimately connected to some semiclassical configurations of the gauge fields, such as electroweak Z-strings and W-condensates. They argued that electroweak strings are formed during the second order electroweak phase transition. This mechanism generates a field $B \sim 10^{-21}$ G on 1 Mpc scale.

Diaz-Gil et al. [28, 29] analyzed the generation of magnetic fields during preheating within a scenario of hybrid inflation at the electroweak scale. They provided a realization of the mechanism proposed by Vachaspati, by which inhomogeneities of the Higgs field phases act as sources for the generation of magnetic fields. Inflation ends at the electroweak scale, then tachyonic preheating develops and non-linearities in the fields cannot be neglected anymore. Gradients in the orientation of the Higgs field create magnetic string-like configurations. The important feature in this scenario is that the induced magnetic fields are helical. During the electroweak symmetry breaking phase the magnetic fields are squeezed in string-like structures between the bubbles. The correlation length grows as fast as the particle horizon and this behaviour is interpreted as an indication that an inverse cascade of magnetic helicity is in operation. However it is not possible to extrapolate the late time behaviour because of the limited knowledge on the primordial plasma features.

In spite of the work done to realize a mechanism for the generation of primordial magnetic fields via first order phase transitions, it seems that within the framework of the standard model a first order electroweak phase transition is incompatible with the Higgs mass experimental lower bound [62].

### 4.6.4 Magnetic fields from the quark-hadron phase transition

In the early Universe at very high temperature a QCD phase transition is predicted, where a quark-gluon plasma condensates into colorless hadrons. Lattice computations suggest that the QCD phase transition is of first order and occurs at $T_{QCD} \sim 150$ MeV [61].

This phase transition is first order so it takes place by bubble nucleation. When the temperature goes below $T_{QCD}$, bubbles containing the hadronic phase grow releasing heat in the quark-gluon plasma. When the shock fronts of these bubbles collide in an out-of-equilibrium process, they reheat the plasma to $T_{QCD}$, stopping the bubble growth. From this moment on new bubbles are nucleated in thermal equilibrium, giving rise to a coexistence phase. The temperature decrease due to the Universe expansion is compensated by the heat released by the bubbles and when the expansion wins over, the transition ends and the remaining pockets of quark-gluon plasma are quickly hadronized.

Quashnock, Loeb and Spergel [106] first proposed the production of magnetic fields via QCD phase transition. The latent heat released by the deflagration front produces a pressure gradient up to the shock front and the gradient generates a radial electric field behind the shock front. This is due to the baryon asymmetry which makes the baryonic component of the primordial plasma positively charged. When the shock fronts collide, a turbulent phase starts and vorticity is generated on the scales of the bubbles. Electric currents circulate on such scales and magnetic fields are generated. The magnetic field generated on scales $10^{10}$ cm, corresponding to 1 AU at present time, is $B \sim 10^{-17}$ G. This small strength is dramatically suppressed if one considers scales of the order of the galactic size $\sim 10$ kpc.

Cheng and Olinto [21] focused on the coexistence phase of the phase transition. They argued that stronger fields can be generated during this phase because of the baryon number susceptibility of the two phases. Magnetic fields are generated by the peculiar motion of the electric dipoles which arises from the convective transfer of the latent heat released by the expanding bubble walls. The generated magnetic field is $B \sim 10^{-16}$ G in 1 pc scales.

However, in [5] Aoki et al. studied the nature of the QCD transition. They proved that the QCD transition in the hot early Universe was not a real phase transition, but an analytic crossover (involving a rapid change, as opposed to a jump, as the temperature varied). Therefore the results for the generation of magnetic fields through the QCD phase transition seem invalid.

### 4.6.5 Magnetic fields from cosmic strings

Cosmic strings are one-dimensional topological defects which formed via the Kibble mechanism during a primordial phase transition. Vachaspati and Vilenkin [121] proposed that cosmic strings may produce plasma vorticity and magnetic fields. Vorticity is produced in the wakes of fast moving cosmic strings after the beginning of structure formation and since the vortical eddies are bounded to the strings, vorticity does not decay with the Universe expansion. The generated field is $B \sim 10^{-18}$ G and the coherence scale is the scale of the wiggles of the string and it can be up to 100 kpc.

Avelino and Shellard [6] proposed an alternative model in which vorticity is produced not by the wiggles, but by the strings themselves, which drag matter behind them because of a finite dynamical friction. However, the field strength predicted is very weak ($B \sim 10^{-23}$ G today).

Witten predicted larger fields in [127], where he considered the case of superconducting cosmic strings. The superconducting charge carriers may be either bosons or fermions. If primordial magnetic fields pre-exist, they may play a role charging up the string loops and delaying their collapse. Otherwise superconducting cosmic strings can generate magnetic fields in a way similar to the one proposed by Avelino and Shellard.

# Chapter 5

# Large-scale magnetic fields and inflation

The origin of magnetic fields is unknown and many scenarios have been proposed to explain them. Until recently the most accepted idea for the formation of large-scale magnetic fields was the exponentiation of a seed field as suggested by Zeldovich and collaborators long ago. This seed mechanism is known as galactic dynamo, the idea is the amplification of a tiny field created early enough by differential rotation of the galaxies and the subsequent generation of the galactic and cluster fields, see Section 4.6.1.

However, recent observational developments have cast serious doubts on this possibility. In fact there are already many reasons to believe that, although this is a possible mechanism in some cases, it cannot be universal [17, 71]. Some of the reasons to think that seeding cannot be an answer are simple [20, 51]. First, the very existence of high $z$ galaxies with fields comparable to the one of the Milky Way is incompatible with the necessary number of turns. Second, the narrowness of the distribution, most galaxies and clusters have fields of a few micro Gauss, and this is not compatible with the different number of rotations and the parameters involved in every galaxy. Furthermore, magnetic fields seem to increase with redshift. Though the evidence is not overwhelming, the sample of Faraday rotations measured is now consistent with an increase and the set includes tens of galaxies showing this pattern. Finally, as pointed out by Dolgov, it is difficult to create the fields in clusters since even the most efficient ejection from point bodies in galaxies like supernovas would have difficulty creating them. All put together, seeding seems to be ruled out and moreover, even if the galactic dynamo was effective, one should justify the presence of a seed field which started the process. This is why the mechanism responsible for the origin of large-scale magnetic fields should be searched in the early Universe [34, 51]. Among the mechanisms for the generation of magnetic fields, some of which we have seen in the previous chapter, inflation seems to be the favourite one, as we will see in the following.

In this chapter we study how inflation can produce large-scale magnetic fields. In Section 5.1 we briefly outline some scenarios of inflationary magnetogenesis. In Section 5.2 we focus on a broad class of these scenarios and study the problem of the back reaction of the generated field on the background.

## 5.1 Production of magnetic fields during inflation

As it was noted by Turner and Widrow in [119], inflation is a prime candidate for the production of magnetic fields for four reasons.

- Inflation provides the means of creating effects on very large scales at very early times, starting from microphysical processes operating on scales smaller than the Hubble radius. If electromagnetic quantum fluctuations are amplified during inflation, they could appear today as large-scale magnetic fields (electric fields should be screened by the high conductivity of the plasma).

- Inflation provides the means to amplify the long-wavelength electromagnetic waves. If the conformal invariance of the electromagnetic field is broken, electromagnetic quantum fluctuations could be excited during the de Sitter expansion. This phenomenon is the analog of the particle production in a rapidly changing spacetime metric.

- During inflation, the Universe is free from charged plasma and is not a good conductor, therefore magnetic flux is not conserved and the ratio of the magnetic field with the radiation energy density, $r = (B^2/8\pi)/\rho_\gamma$, can increase.

- Classical fluctuations with wavelength $\lambda \gtrsim H^{-1}$ of massless, minimally coupled fields can grow *superadiabatically*, i.e. their energy density decreases only as $\sim a^{-2}$, rather than the usual $\sim a^{-4}$.

However, in order to generate magnetic fields during inflation, there is a major problem to overcome. The action for the massless vector field is

$$S = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}d^4x = -\frac{1}{4}\int F_{\mu\rho}F_{\nu\sigma}g^{\mu\nu}g^{\rho\sigma}\sqrt{-g}d^4x, \tag{5.1}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ is conformally invariant. It is easy to see that under conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ the determinant transforms as $g \to \Omega^8 g$ and $g^{\mu\nu} \to \Omega^{-2}g^{\mu\nu}$. This is the reason why in the Friedmann Universe with the metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - \delta_{ik}dx^i dx^k\right), \tag{5.2}$$

the conformal vacuum is preserved. Therefore, if we want to amplify quantum fluctuations on inflation and thus explain the origin of primordial magnetic fields, the conformal invariance of electromagnetism must be broken.

Several ways out this obstacle have been proposed. Turner and Widrow [119] considered the following possibilities. First they investigated the case where the conformal invariance is broken explicitly by introducing a gravitational coupling like

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{b}{2}RA_\mu A^\mu - \frac{c}{2}R_{\mu\nu}A^\mu A^\nu. \tag{5.3}$$

The first term is the usual Maxwell Lagrangian and the other terms are the new interactions which break the conformal invariance and give to the photon a non-zero, time-dependent mass. In fact, one of the most severe constraints to this scenario comes from the experimental upper limit to the photon mass, which is $m_\gamma < 2 \times 10^{-16}$ eV. The authors showed that for some suitable choice of the parameters which enter in the Lagrangian, the strength of the generated magnetic field could be astrophysically interesting, even without invoking the galactic dynamo mechanism. They consider also other possibilities:

(a) the coupling of gravitational and electromagnetic fields through terms of the form $RF^2$:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4m_e^2}(bRF_{\mu\nu}F^{\mu\nu} + cR_{\mu\nu}F^{\mu\kappa}F^{\nu}_{\kappa} + dR_{\mu\nu\lambda\kappa}F^{\mu\nu}F^{\lambda\kappa}), \quad (5.4)$$

where $m_e$ is the electron mass;

(b) a massless, charged scalar field, minimally coupled to both gravity and the elctromagnetic fields:

$$L = -D_\mu\phi(D^\mu\phi)^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (5.5)$$

where $D_\mu = \partial_\mu - ieA_\mu$;

(c) the axion electrodynamics. For energies well below the Pecci-Quinn symmetry-breaking scale $f_a$, the effective Lagrangian for axion electrodynamics is

$$L = -\frac{1}{2}\partial_\mu\theta\partial^\mu\theta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g_a\theta F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad (5.6)$$

where $g_a$ is a coupling constant, $\theta = \phi_a/f_a$, $\phi_a$ is the axion field and $\tilde{F}$ is the dual of $F$.

However, these scenarios do not give an appreciable result.

Another way to produce fields large enough to seed the dynamo is proposed by Dolgov and Silk [33]. They considered the spontaneous breaking of the gauge symmetry of electromagnetism which implies non-conservation of the electric charge.

In [32] Dolgov considers the breaking of conformal invariance due to the trace anomaly, i.e. the trace of the energy-momentum tensor which should be zero in conformally invariant theory becomes non-vanishing due to the triangle diagram connecting two photons to a graviton. This may lead to strong electromagnetic amplification during inflation. In fact the quantum anomaly results in the following modification of the Maxwell equations:

$$\partial_\mu F^\mu_\nu + \kappa\frac{\partial_\mu a}{a}F^\mu_\nu = 0, \quad (5.7)$$

where $a = a(\eta)$ is the scale factor and $\eta$ is the conformal time. The numerical coefficient $\kappa$ in $SU(N)$ gauge theory with $N_f$ charged fermions is equal to $\kappa = \alpha/\pi(11N/3 - 2N_f/3)$. Here $\alpha$ is the fine structure constant at the momentum transfer $p$ equal to the Hubble parameter during inflation $p = H$. Equation (5.7) in the Fourier space gives rise to the equation

$$A'' + k^2 A + \kappa \frac{a'}{a} A' = 0, \tag{5.8}$$

where $A$ is the amplitude of the vector potential. From equation (5.8), Dolgov found that the energy density of the electromagnetic field generated during inflation at the moment when its wave reenters the horizon is $F^2_{\mu\nu}(H\lambda)^\kappa/\lambda^4$ and for $\kappa \sim \mathcal{O}(1)$ the amplitude of the magnetic field can be large enough to seed the observed magnetic fields in galaxies. The additional anomalous term can produce magnetic fields large enough even without a dynamo amplification. The magnitude of the effect is too small in the case of the contribution of one electron loop, but in theories with many charged particles, e.g. grand unified theories, the effect may be significant.

In [118] the authors proposed to break the conformal invariance adding by hand a mass term $m^2 g^{\mu\nu} A_\mu A_\nu/2$ to the Lagrangian. Several models can show this behaviour. First they consider inflation with two scalar fields, $s$ real and $\phi$ complex, where $\phi$ couples minimally to the photon field $A_\mu$, giving it a mass $m_A^2 = 2e^2|\phi|^2$. The potential is $V_s s^2 + m_\phi|\phi|^2/2 + \lambda_\phi|\phi|^4/4 - gs^2|\phi|^2/2$, where $V_s$ is an increasing function. During inflation, $s$ and $|\phi|$ decrease as they roll along the curve $|\phi| = \sqrt{(gs^2 - m_\phi^2)/\lambda_\phi}$ until $s < m_\phi/\sqrt{g}$, after which $|\phi| = 0$. The second possibility is given by the back reaction of the vacuum fluctuations of a scalar field $\langle\Phi^\dagger\Phi\rangle$ on the equation of motion of a minimally coupled gauge field. During inflation $\langle\Phi^\dagger\Phi\rangle$ grows and it would give a mass $2e^2\langle\Phi^\dagger\Phi\rangle$ to the photon. If $\Phi$ decays soon after inflation, the mass goes to zero. This would be the case if $\Phi$ is a heavy squark field or the electroweak Higgs field and the electroweak symmetry is restored by reheating. All these mechanisms result in a gauge field spectrum $A_{\boldsymbol{k}} \sim k^{-1-\nu} \sim k^{-3/2}$, corresponding to a magnetic field $B_l \sim l^{-3/2+\nu} \sim l^{-1}$, where $l$ is the relevant coherence scale and $\nu \simeq 1/2$. This scenario can give rise to a seed field for the dynamo mechanism.

As we discussed above, conformal invariance of the electromagnetism is spoiled if the electromagnetic field is coupled to a scalar field. In [107] Ratra suggested a coupling of the form $e^{\kappa\phi}F^{\mu\nu}F_{\mu\nu}$, where $\kappa$ is an arbitrary parameter. This kind of coupling is produced in some peculiar scenarios of inflation [108] with an exponential inflaton potential. This model may lead to a huge amplification of the electromagnetic quantum fluctuations and the present time intergalactic magnetic field can be as large as $10^{-9}$ G. However, depending on the parameters of the model, the predicted field could also be as low as $10^{-65}$ G.

A slightly more predictive model has been proposed independently by Lemoine and Lemoine [73] and Gasperini, Giovannini and Veneziano [48] and is based on string cosmology. In this scenario the electromagnetic field is coupled not only to the metric $g_{\mu\nu}$, but also to the dilaton field $\phi$. In the low energy limit of the theory and after the dimensional reduction from 10 to 4 spacetime dimensions, such a coupling takes the form $\sqrt{-g}e^{-\phi}F^{\mu\nu}F_{\mu\nu}$, breaks the conformal invariance of the electromagnetic field and coincides with the coupling considered by Ratra [107]. While Ratra assumed that inflation is driven by the scalar field potential, in string cosmology there is the problem that dilaton potentials are too steep to produce the required slow roll of the inflaton field. Thus they assumed that inflation is driven by the kinetic part of the dilaton field. In this scenario the Universe evolves from a flat, cold and weakly coupled ($\phi = -\infty$) initial unstable vacuum state toward a curved, dilaton-driven, strong coupling regime. This period is called pre-Big-Bang phase and is the time when electromagnetic field amplification from vacuum quantum fluctuations takes place. Lemoine and Lemoine [73] estimate that in the most simple model of dilaton-driven inflation a very tiny magnetic field is predicted today. Gasperini et al. [48] claim that larger magnetic fields can be produced on protogalactic scales. This is due to the presence of a new phase between the dilaton-dominated phase and the FRW phase during which the dilaton potential is non-vanishing. The new phase is called string phase and should start when the string length scale $\lambda_s$ becomes comparable to the horizon size at the conformal time $\eta_s$. Unfortunately the duration of this phase is unknown and this makes the model not very predictive.

A more general way to break the conformal invariance is proposed by Bamba and Sasaki [7]. They considered a coupling of the form $IF^2$, where $I$ can be a function of non-trivial background fields that vary in time. The generation of magnetic fields with large amplitude is archieved if the function $I$ is extremely small at the beginning and increases rapidly in time during inflation.

A similar coupling is studied by Martin and Yokoyama in [89]. The inflaton field is described in a supergravity framework where the conformal invariance of the electromagnetic field is naturally broken with a coupling of the form $f^2(\phi)F^{\mu\nu}F_{\mu\nu}$, where $\phi$ is the inflaton field. They determined the form of the coupling that is consistent with the magnetic field observations for different inflationary scenarios. In finding the form of the coupling they also took into account the problem of the back reaction of the magnetic field on the background. Then they studied whether the required coupling can naturally emerge in well-motivated models, but they realized that this is nontrivial and can be realized only for a restricted class of scenarios, among which power-law inflation. However, this scenario seems consistent only if the energy scale of inflation is low and the reheating stage is prolonged.

An interesting possibility was studied by Durrer, Hollenstein and Jain in [35]. They considered the generation of helical magnetic fields during single-field inflation, where a coupling to the parity-violating term $\tilde{F}F$, i.e. a term $f(\phi)\tilde{F}F$, is added to the standard electromagnetic action $F^2$, where $\tilde{F}$ is the dual of $F$ and $\phi$ is the inflaton. As a consequence magnetic helicity is generated. This has two effects. Helical magnetic fields evolve in the cosmic magnetohydrodynamic plasma via inverse cascade and this transfers power from small to large scales, so that even blue spectra can lead to significant power on large scales. Second, since helical magnetic fields violate parity symmetry, they leave a very distinctive signature and therefore observable effects, e.g. correlations between the anisotropies in the temperature and B-polarization, or in the E- and the B-polarizations in the CMB. They showed that a helical coupling always leads to a spectral index $n = 1$ for $B^2(k) \sim k^n$, as long as slow-roll inflation is considered. Even though the inverse cascade in the radiation dominated era after inflation moves power to larger scales, the final strength of the magnetic field on cosmologically interesting scales is still insufficient to provide seeds for the observed magnetic fields in galaxies and clusters, except if the inflation scale is low, $T_* < 10^4$ Gev and the axial coupling is very strong.

## 5.2 Back reaction of the generated magnetic fields

In this section we follow our paper [27]. We consider a broad class of models where conformal invariance is broken during inflation and investigate the back reaction of the generated magnetic field on the background. We show that the back reaction is very important and leads to rather strong bounds on the maximal value of the strength of primordial magnetic fields which seems not enough to explain the observed fields as a result of amplification of these primordial seeds by dynamo mechanism.

In the previous section we have seen that if we want to explain the origin of primordial magnetic fields via amplification of quantum fluctuations on inflation, the conformal invariance of electromagnetism must be broken. Most of the models considered above are effectively reduced either to the appearance of an effective mass or a time dependent coupling constant. Both of these options are taken into account if we write the action in the form

$$S = \int \left( -\frac{1}{4} I^2 F_{\mu\nu} F^{\mu\nu} + M^2 A_\mu A^\mu \right) \sqrt{-g} d^4x. \tag{5.9}$$

Here $I(t) = I(\phi(t), \cdots)$, where $\phi$ can be the inflaton, dilaton or some other scalar field and the dots can be anything, for instance, invariants of the curvature [7, 48, 107]. The appearance of the time dependence of the coefficient in front of $F^2$ term is naturally interpreted as a time-dependent coupling constant of the vector field. In fact if we write the Lagrangian density of the vector field coupled with a charged fermion in the standard form as

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + igA_\mu)\psi, \tag{5.10}$$

where $g$ is the coupling constant, then after rescaling the vector potential by the coupling constant $A_\mu \to gA_\mu$ we bring this Lagrangian to the form

$$L = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + iA_\mu)\psi, \tag{5.11}$$

which is "ready" for introducing a time-dependent coupling constant. Note that $I$ is an inverse coupling constant and small values of $I$ correspond to a large coupling constant $g$, which in turn would mean that we are in an uncontrollably strong coupling regime. Only if $I$ is large we can trust the theory. For our purposes we do not need to specify in more detail the origin of the time dependence of $I$ here. Note that the time-dependent effective coupling leaves the Lagrangian to be $U(1)$ gauge-invariant.

The mass term introduced by "hand" spoils gauge invariance. Only when it is generated via Higgs mechanism the gauge invariance is preserved. On the other hand as it was noticed already in [119], large enough magnetic fields can be obtained only if $M^2$ is negative during inflation. However, to generate negative mass squared term via Higgs mechanism one needs a ghost scalar field with negative kinetic energy [37, 57]. As it is well known, ghosts lead to catastrophic instabilities and therefore we will not exploit this possibility any further here. Instead we introduce the effective negative mass square terms considering the non-minimal coupling of the vector field to gravity, so that,

$$M^2 = m^2 + \xi R, \tag{5.12}$$

where for generality we also keep the "hard" mass term $m^2$, assuming that it is positive.

Let us now rewrite the action (5.9) in terms of the vector potential $A_\alpha = (A_0, A_i)$. It is convenient to decompose the spatial part of the vector potential in terms of its transverse and longitudinal components $A_i = A_i^T + \partial_i \chi$, where $\partial_i A_i^T = 0$ (we will be assuming summation over repeated indices irrespective of their position). In the homogeneous flat Universe with metric (5.2), the action (5.9) then becomes

$$\begin{aligned} S = \frac{1}{2} \int [ & I^2 \left( A_i^{T\prime} A_i^{T\prime} + A_i^T \Delta A_i^T + 2A_0 \Delta \chi' - A_0 \Delta A_0 - \chi' \Delta \chi' \right) \\ & + M^2 a^2 \left( A_0^2 + \chi \Delta \chi - A_i^T A_i^T \right) ] d^4 x, \end{aligned} \tag{5.13}$$

where prime denotes the derivative with respect to the conformal time $\eta$. We will consider different cases separately.

### 5.2.1 Time dependent coupling

Let us first consider the case when $M^2 = 0$ and $I = I(t)$. Then the variation of the action (5.13) with respect to $A_0$ gives $A_0 = \chi'$, and the action simplifies to

$$S = \frac{1}{2}\int I^2 \left(A_i^{T\prime} A_i^{T\prime} + A_i^T \Delta A_i^T\right) d^4x. \tag{5.14}$$

Substituting the expansion

$$A_i^T(\boldsymbol{x},\eta) = \sum_{\sigma=1,2}\int A_{\boldsymbol{k}}^{(\sigma)}(\eta)\varepsilon_i^{(\sigma)}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{d^3k}{(2\pi)^{3/2}}, \tag{5.15}$$

where $\varepsilon_i^{(\sigma)}(\boldsymbol{k})$, $\sigma = 1,2$ are two orthogonal polarization vectors, into (5.14), we obtain

$$S = \frac{1}{2}\sum_{\sigma=1,2}\int I^2\varepsilon_i^{(\sigma)}(\boldsymbol{k})\varepsilon_i^{(\sigma)}(-\boldsymbol{k})\left(A_{\boldsymbol{k}}^{(\sigma)\prime}A_{-\boldsymbol{k}}^{(\sigma)\prime} - k^2 A_{\boldsymbol{k}}^{(\sigma)}A_{-\boldsymbol{k}}^{(\sigma)}\right) d\eta d^3k. \tag{5.16}$$

Rewritten in terms of the new variable

$$v_{\boldsymbol{k}}^{(\sigma)} = \sqrt{\varepsilon_i^{(\sigma)}\varepsilon_i^{(\sigma)}}\, I A_{\boldsymbol{k}}^{(\sigma)}, \tag{5.17}$$

this action becomes

$$S = \frac{1}{2}\sum_{\sigma=1,2}\int \left(v_{\boldsymbol{k}}^{(\sigma)\prime}v_{-\boldsymbol{k}}^{(\sigma)\prime} - \left(k^2 - \frac{I''}{I}\right)v_{\boldsymbol{k}}^{(\sigma)}v_{-\boldsymbol{k}}^{(\sigma)}\right) d\eta d^3k. \tag{5.18}$$

It describes two real scalar fields with time-dependent effective masses in terms of their Fourier components.

We are interested in the correlation functions of the transverse components of the vector potential and magnetic field assuming that initially the field was in its vacuum state. The quantization of the fields with action (5.18) is standard and we will simply summarize here the results referring the reader to [93, 94] for the details. Taking into account (5.17) and (5.15), we immediately find the correlation function

$$< 0|\hat{A}_i^T(\eta, \boldsymbol{x})\hat{A}^{Ti}(\eta, \boldsymbol{y})|0 >= -\frac{1}{4\pi^2 a^2 I^2}\sum_{\sigma=1,2}\int |v_{\boldsymbol{k}}^{(\sigma)}(\eta)|^2 k^3 \frac{\sin k|\boldsymbol{x}-\boldsymbol{y}|}{k|\boldsymbol{x}-\boldsymbol{y}|}\frac{dk}{k}, \tag{5.19}$$

where $v_{\boldsymbol{k}}^{(\sigma)}(\eta)$ satisfy the equations

$$v_{\boldsymbol{k}}^{(\sigma)\prime\prime} + \omega^2(\eta)\, v_{\boldsymbol{k}}^{(\sigma)} = 0, \quad \omega^2(\eta) \equiv \left(k^2 - \frac{I''}{I}\right), \tag{5.20}$$

which immediately follow from action (5.18). The initial conditions for these equations corresponding to the initial vacuum state at $\eta_i$ are

$$v_{\boldsymbol{k}}^{(\sigma)}(\eta_i) = \frac{1}{\sqrt{\omega(\eta_i)}}, \quad v_{\boldsymbol{k}}^{(\sigma)\prime}(\eta_i) = i\sqrt{\omega(\eta_i)}. \tag{5.21}$$

These initial conditions make sense only if $\omega^2 > 0$. Anyway, we will need them only for the short-wavelength modes for which $\omega^2 \simeq k^2$. The power spectrum characterizing the typical amplitude squared of the invariant magnitude of the vector potential, $A = \sqrt{-A_i A^i}$, in the appropriate comoving scale $\lambda = 2\pi/k$ is

$$\delta_A^2(k, \eta) = \sum_{\sigma=1,2}\frac{|v_{\boldsymbol{k}}^{(\sigma)}(\eta)|^2 k^3}{4\pi^2 a^2 I^2}. \tag{5.22}$$

Taking into account that the magnitude of the magnetic field is

$$B^2 = -B_i B^i = \frac{1}{2a^4}F_{ik}F_{ik} = \frac{1}{a^4}\left(\partial_i A_k \partial_i A_k - \partial_k A_i \partial_i A_k\right), \tag{5.23}$$

we obtain for the power spectrum of the magnetic field

$$\delta_B^2(k, \eta) = \delta_A^2(k, \eta)\frac{k^2}{a^2} = \sum_{\sigma=1,2}\frac{|v_{\boldsymbol{k}}^{(\sigma)}(\eta)|^2 k^5}{4\pi^2 a^4 I^2}, \tag{5.24}$$

that is, its amplitude decays faster by an extra power of the scale compared to the amplitude of the vector potential. For example, a flat spectrum of the magnetic field ($\delta_B(k) = const$) corresponds to the linearly growing towards large scales spectrum for the vector potential, that is, $\delta_A(k, \eta) \propto k^{-1}$.

We will need to control the back reaction of the generated electromagnetic field on the background. For this purpose let us calculate the expectation value of the energy density equal to $T_0^0$ component of the energy-momentum tensor:

$$T_0^0 = I^2 \left( \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - F_{0\alpha} F^{0\alpha} \right) = \frac{I^2}{2a^4} \left( A_i^{T\prime} A_i^{T\prime} + \partial_i A_k^T \partial_i A_k^T \right). \tag{5.25}$$

Taking into account (5.15) and (5.17) we obtain

$$\begin{aligned} \varepsilon_{EM} &= < 0|\hat{T}_0^0|0 > \qquad (5.26) \\ &= \frac{1}{8\pi^2 a^4} \sum_{\sigma=1,2} \int \left[ |v_k^{(\sigma)\prime}(\eta)|^2 - \frac{I'}{I} |v_k^{(\sigma)}(\eta)|^{2\prime} + \left( \frac{I'^2}{I^2} + k^2 \right) |v_k^{(\sigma)}(\eta)|^2 \right] k^3 \frac{dk}{k}. \end{aligned}$$

Let us assume that the function $I$ depends on time during inflation and find the resulting spectrum of the magnetic field at the end of inflation. For short waves with $k\,|\eta| \gg 1$ we can neglect $I''/I$ compared to $k^2$ in (5.20) and the solution of this equation with vacuum initial conditions (5.21) then becomes

$$v_k^{(\sigma)}(\eta) \simeq \frac{1}{\sqrt{k}} e^{ik(\eta - \eta_i)}. \tag{5.27}$$

Because $|\eta|$ decreases during inflation, at some moment $|\eta_k| \simeq 1/k$ the physical scale of the wave with comoving wavenumber $k$ begins to exceed the curvature scale and taking into account that $k^2 \ll I''/I$, we can write the general longwave solution of (5.20) as

$$v_k^{(\sigma)}(\eta) \simeq C_1 I + C_2 I \int \frac{d\eta}{I^2}, \tag{5.28}$$

where $C_1$ and $C_2$ are the constants of integration which have to be fixed by matching this solution to (5.27) at $|\eta_k| \simeq 1/k$. Let us assume that $I$ is a power-law function of the scale factor during inflation

$$I = I_f \left( \frac{a}{a_f} \right)^n, \tag{5.29}$$

where $a_f$ is the scale factor at the end of inflation. Taking into account that

$$d\eta = \frac{da}{Ha^2}, \tag{5.30}$$

and the Hubble constant $H$ does not change significantly during inflation, we obtain from (5.28)

$$v_k^{(\sigma)}(\eta) \simeq C_1 a^n + C_2 a^{-n-1}. \tag{5.31}$$

#### 5.2.1.1 Strong coupling case

In the case $n > -1/2$ the first mode dominates and, matching solutions (5.27) and (5.31) at $|\eta_k| \simeq 1/k$, we find

$$v_k^{(\sigma)}(\eta) \simeq \frac{1}{\sqrt{k}} \left(\frac{a}{a_k}\right)^n \simeq \frac{1}{\sqrt{k}} \left(\frac{H_I a}{k}\right)^n, \tag{5.32}$$

where we have taken into account that at the moment $\eta_k$, when the corresponding wave crosses the Hubble scale, the scale factor is $a_k \simeq k/H_I$. Substituting (5.32) into (5.24) we obtain at the end of inflation

$$\delta_B(\lambda_{ph}, \eta_f) \simeq \frac{H_I^2}{\sqrt{2\pi} I_f} \left(\frac{\lambda_{ph}}{H_I^{-1}}\right)^{n-2}, \tag{5.33}$$

where $\lambda_{ph} = a_f/k$ is the physical wavelength and $H_I$ is the Hubble constant on inflation. This formula is valid for $H_I^{-1}(a_f/a_i) > \lambda_{ph} > H_I^{-1}$, where $a_i$ is the value of the scale factor at the beginning of inflation. If $n = 2$, the spectrum of the magnetic field is flat. For $H_I^2 \simeq 10^{-12}$ (in Planck units), required by primordial inhomogeneities [93], and $I_f \simeq \mathcal{O}(1)$, the amplitude of the field is the same in all scales and it is equal to $\delta_B \simeq 10^{-12}$ Planck units or $\sim 10^{46}$ G immediately after inflation. Later on the magnetic field is frozen and decays inversely proportional to the scale factor squared. To estimate how much the scale factor increases after inflation, we can use the entropy conservation law. Assuming that inflation is followed by the dust dominated stage we obtain

$$\frac{a_0}{a_f} \simeq g^{1/12} \frac{H_I^{1/2}}{T_0} \left(\frac{a_R}{a_f}\right)^{1/4}, \tag{5.34}$$

where $g$ is the number of relativistic degrees of freedom of those particles which later on transfer their entropy to the photons, $T_0$ is the temperature of the background radiation today and $a_R$ the scale factor at the moment of reheating. The lower bound on this ratio is obtained by assuming that reheating happens immediately after inflation. In this case for $H_I \simeq 10^{-6}$ we have $a_0/a_f \simeq 10^{29}$ and correspondingly the strength of the generated magnetic field cannot exceed $10^{-12}$ $G$.

Let us calculate the energy density of the generated magnetic field. The main contribution to the energy density comes from the scales exceeding $H_I^{-1}$ because the contribution from the subhorizon scales is renormalized in the leading order. In the case where the dominant mode $v \propto I$ and $A^T \propto v/I \propto const$, the time derivatives of the vector potential in (5.25) contribute only in subleading $k^2$ order and their contribution is comparable to the contribution of the magnetic field itself given by the last term in (5.26). Thus we obtain

$$\varepsilon_{EM} = \frac{\mathcal{O}\left(1\right)}{a^4}\int_{H_I a_i}^{H_I a} |v_k(\eta)|^2 k^4 dk, \tag{5.35}$$

where $a_i$ is the value of the scale factor at the beginning of inflation. Substituting (5.32) into (5.35) we find that at the end of inflation when $a = a_f$

$$\varepsilon_{EM} = \mathcal{O}\left(1\right) H_I^4 \times \left\{ \begin{array}{ccc} \frac{1}{2-n}, & & n < 2, \\ \ln\left(\frac{a_f}{a_i}\right), & \text{for} & n = 2, \\ \frac{1}{n-2}\left(\frac{a_f}{a_i}\right)^{2(n-2)}, & & n > 2. \end{array} \right. \tag{5.36}$$

We see that the magnetic field energy can be comparable with the energy density of the background only for $n \geq 2$. Requiring that inflation should last at least 75 e-folds, we obtain that the contribution of the magnetic field energy density does not spoil inflation, that is, $\varepsilon_{EM}$ is smaller that $H_I^2$ until the end of inflation, only if $n - 2 < 0.2$. Thus, we can have a magnetic field spectrum which is slightly growing toward large scales. In particular, for $n \simeq 2.2$ the amplitude of the magnetic field in Mpc scales can be larger by a factor $10^5$ compared to the above considered case of the flat spectrum, that is, $\delta_B \simeq 10^{-7}$ G today. This is the greatest amplitude of the primordial magnetic field which we can obtain in the above considered case. Note that the theory where $I$ grows with the scale factor corresponds to the case where the effective coupling constant, which is inversely proportional to $I$, is incredibly large at the beginning of inflation and becomes of the order of one at the end of inflation. Hence at the beginning we are in a strongly coupled regime where such a theory is not trustable at all.

The case considered above is the only one in which we can generate strong enough fields on inflation. Let us show that in all other cases there is very strong bound on the possible value of the generated field due to the back reaction of this field on the background.

#### 5.2.1.2 Weak coupling case

For $n < -1/2$ the second term in (5.31) dominates and

$$v_k\left(\eta\right) \propto a^{-n-1}. \tag{5.37}$$

In this case the result follows immediately by substituting in the formulae (5.32) and (5.33) $-n-1$ instead of $n$, so that

$$v_k^{(\sigma)}\left(\eta\right) \simeq \frac{1}{\sqrt{k}}\left(\frac{a}{a_k}\right)^{-n-1} \simeq \frac{1}{\sqrt{k}}\left(\frac{H_I a}{k}\right)^{-n-1}, \tag{5.38}$$

and

$$\delta_B\left(\lambda_{ph}, \eta_f\right) \simeq \frac{H_I^2}{\sqrt{2}\pi I_f}\left(\frac{\lambda_{ph}}{H_I^{-1}}\right)^{-n-3}. \tag{5.39}$$

Thus the spectrum of the magnetic field is flat for $n = -3$. This case corresponds to the coupling constant growing as $I^{-1} \propto a^3$, that is, it changes from extremely small values at the beginning of inflation to values of order of unity at the end of inflation. Thus the theory is trustable everywhere. However, here the back reaction of the field is very large because $A \propto v/I \propto a^{-2n-1}$ changes very fast and the main contribution to the energy density comes from the time derivative of the vector potential in (5.25), that is, from the electric field. Substituting (5.38) in (5.26) we obtain that at the end of inflation

$$\varepsilon_{EM} \simeq \frac{4n^2+4n+1}{8\pi^2} H_I^4 \times \begin{cases} \frac{1}{n+2}, & n > -2, \\ \ln\left(\frac{a_f}{a_i}\right), & \text{for} \quad n = -2, \\ -\frac{1}{n+2}\left(\frac{a_f}{a_i}\right)^{-2(n+2)}, & n < -2. \end{cases} \tag{5.40}$$

Requiring that inflation should last at least 75 e-folds, we find that $\varepsilon_{EM} < H_I^2$ at the end of inflation only if $n > -2.2$. Thus the flat spectrum of the magnetic field cannot be generated during inflation because in this case the back reaction of the electromagnetic field would spoil inflation too early. In the most favorable admissible case $n \simeq -2.2$, the amplitude of the magnetic field decays as $\delta_B \propto \lambda_{ph}^{-0.8}$ and its value cannot exceed $10^{-32}$ G in Mpc scales today. Thus in this model with weak coupling constant during inflation one cannot explain the origin of the primordial magnetic field.

### 5.2.2 Massive field

Now we set $I = 1$ and consider the case where magnetic fields are generated by the mass term in the action. Variation of action (5.13) with respect to $A_0$ gives

$$\Delta\chi' - \Delta A_0 + M^2 a^2 A_0 = 0. \tag{5.41}$$

Taking the Fourier transform

$$\chi(\boldsymbol{x},\eta) = \int \chi_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{d^3k}{(2\pi)^{3/2}}, \quad A_0(\boldsymbol{x},\eta) = \int A_{0\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{d^3k}{(2\pi)^{3/2}}, \tag{5.42}$$

we obtain from here

$$A_{0\boldsymbol{k}} = \frac{k^2}{k^2 + M^2 a^2} \chi'_{\boldsymbol{k}} \equiv F_k \chi'_{\boldsymbol{k}}. \tag{5.43}$$

Substituting into the action (5.13) the expansions (5.15), (5.42) and using (5.43) to express $A_{0\boldsymbol{k}}$ in terms $\chi'_{\boldsymbol{k}}$ we obtain

$$S = \frac{1}{2}\sum_{\sigma=1,2}\int\left(v_{\boldsymbol{k}}^{(\sigma)\prime}v_{-\boldsymbol{k}}^{(\sigma)\prime} - \left(k^2+M^2a^2\right)v_{\boldsymbol{k}}^{(\sigma)}v_{-\boldsymbol{k}}^{(\sigma)}\right)d\eta d^3k \tag{5.44}$$
$$+\frac{1}{2}\int sign\left(1-F_k\right)\left(\bar{\chi}'_{\boldsymbol{k}}\bar{\chi}'_{-\boldsymbol{k}} - \left(k^2+M^2a^2-\frac{\sqrt{|1-F_k|}''}{\sqrt{|1-F_k|}}\right)\bar{\chi}_{\boldsymbol{k}}\bar{\chi}_{-\boldsymbol{k}}\right)d\eta d^3k,$$

where $v_{\boldsymbol{k}}^{(\sigma)}$ is defined in (5.17) ($I=1$), and

$$\bar{\chi}_{\boldsymbol{k}} = k\sqrt{|1-F_k|}\chi_{\boldsymbol{k}}. \tag{5.45}$$

Thus we see that in the case of massive field the longitudinal degree of freedom $\chi$ becomes dynamical. In the case of positive mass squared, $F_k$ is always smaller than unity and therefore the sign in front of the longitudinal part of the action is positive. However, if $M^2$ is negative then $1-F_k$ is negative for high momentum modes with $k^2 > M^2a^2$ and these modes have negative kinetic energy. The low momentum modes with $k^2 < M^2a^2$ have positive kinetic energy because $F_k$ is negative for them. Thus, introducing a tachyonic mass for the vector field in a "hard" way seems to lead inevitably to the appearance of ghost for high momentum longitudinal modes [37]. Therefore if we want to avoid catastrophic instabilities related with ghosts fields we have to consider a tachyonic vector field only as a low energy effective field theory description of some yet unknown theory with "safe" ultraviolet completion. On the other hand if negative effective mass appears as an interaction with the curvature, $M^2 = \xi R$, then the field is massless on scales smaller than the typical distance between particles inducing the average curvature and thus there is a natural ultraviolet cutoff in the theory. Note that this argument is not directly applicable in the presence of the cosmological constant. Let us assume that the problem of ghosts can be somehow solved and proceed with the calculation of the magnetic field from inflation in the theory with $M^2 = m^2 + \xi R$. In the case $m = 0$ the photon mass is $m_\gamma \sim R^{1/2}$, where $R^{1/2} \sim H$. Today it would be $m_\gamma = H_{today} \sim 10^{-33}eV$, well below the available experimental limits on the photon mass. The breaking of charge conservation also manifests itself only on scales of the horizon or larger ($\geq H^{-1} \sim 10^{28} cm$) and hence has no observable consequences.

The equations of motion for transverse and longitudinal modes follow immediately from the action (5.44):

$$v_{\boldsymbol{k}}^{(\sigma)\prime\prime} + \left(k^2 + M^2 a^2\right) v_{\boldsymbol{k}}^{(\sigma)} = 0, \tag{5.46}$$

and

$$\bar{\chi}_{\boldsymbol{k}}^{\prime\prime} + \left(k^2 + M^2 a^2 - \frac{\sqrt{|1 - F_k|}^{\prime\prime}}{\sqrt{|1 - F_k|}}\right) \bar{\chi}_{\boldsymbol{k}} = 0. \tag{5.47}$$

Let us consider the de Sitter Universe where

$$a = -\frac{1}{H_I \eta}. \tag{5.48}$$

Taking into account that $R = -12 H_I^2$, for $m^2 = 0$ equation (5.46) becomes

$$v_{\boldsymbol{k}}^{(\sigma)\prime\prime} + \left(k^2 - \frac{12\xi}{\eta^2}\right) v_{\boldsymbol{k}}^{(\sigma)} = 0. \tag{5.49}$$

For short waves with $k\,|\eta| \gg 1$, the solution of this equation corresponding to vacuum initial conditions is

$$v_k^{(\sigma)}(\eta) \simeq \frac{1}{\sqrt{k}} e^{ik(\eta - \eta_i)}. \tag{5.50}$$

For $k\,|\eta| \ll 1$ we can neglect the $k^2$ term in (5.49), and the dominating long-wavelength solution of this equation is

$$v_k^{(\sigma)}(\eta) \simeq \frac{1}{\sqrt{k}} \left(\frac{H_I a}{k}\right)^n, \quad n = \frac{1}{2}\left(\sqrt{1 + 48\xi} - 1\right), \tag{5.51}$$

where we use the matching conditions at $|\eta_k| \simeq 1/k$ to fix the constant of integration. Since here the calculations are very similar to those in the previous section we can immediately write the result for the magnetic field:

$$\delta_B\left(\lambda_{ph}, \eta_f\right) \simeq \mathcal{O}(1)\, H_I^2 \left(\frac{\lambda_{ph}}{H_I^{-1}}\right)^{n-2}. \tag{5.52}$$

For $\xi = 1/6$ we have $n = 1$ and the spectrum linearly decays with the scale. In this case its value today is about $10^{-37}$ G in Mpc scales. The flat spectrum is obtained for $\xi = 1/2$. However, to find out whether this case is possible, we have to verify that the back reaction of the magnetic field will not spoil inflation too early. In the energy density also contributes the longitudinal mode and to determine its contribution we will need a long-wavelength solution for $\bar{\chi}_{\boldsymbol{k}}$. It is easy to check that the term which is different in the equations (5.46) and (5.47) can be neglected for both short-wave and long-wave solutions and hence

$$\bar{\chi}_k(\eta) \simeq \frac{1}{\sqrt{k}}\left(\frac{H_I a}{k}\right)^n, \quad n = \frac{1}{2}\left(\sqrt{1+48\xi}-1\right). \tag{5.53}$$

Variation of action (5.9), where $I = 1$ and $M^2 = m^2 + \xi R$, with respect to the metric gives

$$\begin{aligned} T^{\rho}_{\mu} = &\frac{1}{4}\delta^{\rho}_{\mu}F_{\alpha\beta}F^{\alpha\beta} - F^{\rho\beta}F_{\mu\beta} - \frac{1}{2}\delta^{\rho}_{\mu}(m^2+\xi R)A_{\alpha}A^{\alpha} \\ &+ (m^2+\xi R)A_{\mu}A^{\rho} + \xi R^{\rho}_{\mu}A_{\alpha}A^{\alpha} + \xi[\delta^{\rho}_{\mu}\nabla^{\alpha}\nabla_{\alpha}(A^{\beta}A_{\beta}) - \nabla_{\mu}\nabla^{\rho}(A^{\beta}A_{\beta})]. \end{aligned} \tag{5.54}$$

As a result of straightforward but rather lengthy calculations we obtain

$$< 0|\hat{T}^0_0|0 >= \varepsilon_T + \varepsilon_L, \tag{5.55}$$

where

$$\varepsilon_T = \frac{1}{8\pi^2 a^4}\sum_{\sigma=1,2}\int \left[|v^{(\sigma)\prime}_k|^2 - 6\xi aH|v^{(\sigma)}_k|^{2\prime} + \left(k^2+m^2a^2+6\xi H^2a^2\right)|v^{(\sigma)}_k|^2\right]k^3\frac{dk}{k} \tag{5.56}$$

is the contribution of the transverse modes and $H = a'/a^2$ is the Hubble constant. The contribution of the longitudinal mode is given by

$$\begin{aligned} \varepsilon_L = &\frac{1}{8\pi^2 a^4}\int (1-F)\left\{(1-6\xi bF)|\tilde{\chi}'_k|^2 - 6\xi aH\left(\frac{1+F}{1-F}\right)|\tilde{\chi}_k|^{2\prime}\right. \\ &\left. + \left(\frac{m^2a^2+6\xi H^2a^2}{1-F}\right)|\tilde{\chi}_k|^2\right\}k^3\frac{dk}{k}, \end{aligned} \tag{5.57}$$

where

$$\tilde{\chi} = \bar{\chi}/\sqrt{|1 - F_k|}, \quad b = \frac{\dot{H} + 7H^2 + 4H\frac{\dot{M}}{M}}{M^2}. \tag{5.58}$$

For the longwave modes with $k^2 \ll |M^2 a^2|$ we have $F_k \ll 1$, $\tilde{\chi} \simeq \bar{\chi}$ and their contribution to the total energy density is the same as the contribution of the transverse mode. It is interesting to note that the longitudinal mode is the ghost in the de Sitter background. However, in the Friedmann Universe filled by matter with positive pressure it is not ghost in spite of the fact that the effective mass squared is negative.

Substituting (5.51) into (5.56) we find that in the leading order the contribution of the long-wave modes into the energy density in the case $m = 0$ is

$$\varepsilon_L \simeq \mathcal{O}(1) \frac{H_I^2}{a^2} \left(n^2 - 12n\xi + 6\xi\right) \int_{H_I a_i}^{H_I a} |v_k(\eta)|^2 k^2 dk, \tag{5.59}$$

and calculating the integral we obtain

$$\varepsilon_L \simeq \mathcal{O}(1) H_I^4 \left(n^2 - 12n\xi + 6\xi\right) \begin{cases} \frac{1}{1-n}, & \text{for} \quad n < 1, \\ \frac{1}{n-1}\left(\frac{a}{a_i}\right)^{2(n-1)}, & \text{for} \quad n > 1. \end{cases} \tag{5.60}$$

In the case $\xi = 1/6$ and when $n = 1$ the contribution is canceled in the leading order and $k^2$ terms give a contribution of the order of $H_I^4$, which is the same as for $n < 1$. However, for $\xi > 1/6$, and correspondingly $n > 1$, the energy density of the long-wavelengh electromagnetic waves grows with time rather fast. It is negative and therefore when it becomes of order $H_I^2$ inflation is over. Requiring that inflation should last at least 75 e-folds we find that the contribution of electromagnetic field does not spoil inflation only if $n - 1 < 0.2$. Thus, in the most favorable case of $n \simeq 1.2$, the amplitude of the magnetic field decays as $\delta_B \propto \lambda_{ph}^{-0.8}$ and its value does not exceed $10^{-32}$ G in Mpc scales today.

### 5.2.3 Conclusions

We have studied the generation of large-scale magnetic fields in two classes of models. In the first case the conformal invariance of the Maxwell field is broken by a non-minimal coupling of the form $RA^2$, which gives a non-zero time-dependent mass to the photon. In the second case the conformal invariance is violated because of the time-dependent coupling constant, $I(t)F^{\mu\nu}F_{\mu\nu}$, where $I(t) = I(\phi(t), \cdots)$ is a general function of nontrivial background fields and $\phi$ can be for instance inflaton or dilaton.

In principle it looks like inflation can strongly amplify the vacuum quantum fluctuations and therefore can lead to sizable magnetic fields. However, if we take into account the back reaction of the electromagnetic field and require that inflation lasts at least 75 e-folds, the strength of the primordial field cannot exceed $10^{-32}$ G on Mpc scales and it is not clear whether such a small field can work as a seed for a possible dynamo mechanism.

Only in the strong coupling case, $I(t)F^{\mu\nu}F_{\mu\nu}$, where $I = I_f\,(a/a_f)^n$ and $n \simeq 2.2$, the amplitude can reach the interesting value of $10^{-7}$ G today. However, this case corresponds to the situation where the effective coupling constant is extremely large at the beginning of inflation and becomes of the order of one at the end of inflation and hence the theory is not trustable.
We conclude therefore that the models considered above are not efficient in producing primordial magnetic fields during inflation and, even if the galactic dynamo was effective, the field produced seems to be too small to play the role of a seed for this mechanism.

# Chapter 6

# Conclusions

The success of the inflationary paradigm in describing the observed properties of the Universe is outstanding. We have seen how inflation can explain the amplification of the primordial perturbations, which give rise to all cosmological structures, and the temperature anisotropies in the CMB, which we observe today through many experiments. We have shown that so far the simplest model of inflation, a single-field slow-roll scenario, is in perfect agreement with the observations. In fact it predicts Gaussian curvature perturbations with an almost scale-invariant power spectrum. The 7-year WMAP analysis confirms these predictions, but it leaves an open question about the issue of a possible non-Gaussianity of the perturbations. Indeed the value of the non-linearity parameter $f_{NL}$ is found to be within the range $-10 < f_{NL}^{local} < 74$ [68]. Therefore inflationary theories which provide a higher level of non-Gaussianity still fit the data. Many models that can enhance the level of the primordial non-Gaussianity have been proposed in the literature. Among them, a simple deviation from the standard scenario is represented by a theory of inflation with two scalar fields, called the curvaton scenario.

We have studied the curvaton scenario, describing the generation of curvature perturbations and calculating the level of non-Gaussianity. Then we have discussed the realization of the curvaton scenario in supergravity in the context of chaotic inflation. We have shown that the observational consequences of the resulting scenario, which we called supercurvaton, are very interesting. In fact we have computed the level of non-Gaussianity and we have found that the

$f_{NL}$ parameter is in the observationally interesting range from $\mathcal{O}(10)$ to $\mathcal{O}(100)$. Moreover, our investigation has demonstrated that if inflation is long enough, then the average value of the curvaton contribution to the amplitude of metric perturbations, as well as the averaged value of the parameter $f_{NL}$, do not depend on the initial conditions for the curvaton field. The final results depend on the inflaton mass $m$, and on the parameter $\alpha$. Therefore, while the curvaton models are more complicated than the single-field inflationary models, they make the resulting scenario much more flexible, which may be important for a proper interpretation of the coming observational data.

Further, we have introduced the issue of the ubiquitous presence of large-scale magnetic fields in the Universe, outlining the basic effects on the CMB and the constraints coming from Big Bang nucleosynthesis. Then we have presented various mechanisms for magnetogenesis in the early Universe.

In particular we have focussed on the study of inflationary magnetic fields. Inflation can amplify quantum fluctuations giving rise to long-wavelength magnetic fields. The necessary condition is that the conformal invariance of electromagnetism is broken. In principle it looks like inflation can strongly amplify the magnetic fields. However, inflationary magnetogenesis is not problem free. In fact the generated magnetic field might back react on the background spoiling the inflationary stage. We have briefly reviewed different proposals for the generation of magnetic fields during inflation and we have noted that in the majority of the models the breaking of the conformal invariance is effectively reduced either to the appearance of an effective mass or a time dependent coupling constant. In the first case the conformal invariance is broken by a non-minimal coupling of the form $RA^2$ and in the second one because of the time-dependent coupling constant of the form $I(t)F^2$, where $I(t) = I(\phi(t), \cdots)$ is a general function of non-trivial background fields. We have studied these two broad classes of models and provided limits on the generated magnetic fields by taking into account the back reaction of the electromagnetic field and by requiring that inflation lasts at least 75 e-folds. The result is that the strength of the primordial field cannot exceed $10^{-32}$ G on Mpc scales. The only case where the amplitude of the field can reach the interesting value $10^{-7}$ G is in the theory $I(t)F^2$ with $I = (a/a_f)^n$

and $n \simeq 2.2$. However, this case corresponds to the situation where the effective coupling constant, which is inversely proportional to $I$, is extremely large at the beginning of inflation and approaches unity at the end of inflation. Hence at the beginning we are in a strongly coupled regime and the theory is not trustable. We have concluded that the two broad classes of models we have considered are very much constrained by requiring that the back reaction of the generated magnetic field on the background evolution is small. The back reaction leads to strong bounds on the maximal value of the field strength, which is not enough to explain the observed magnetic fields, even if the dynamo mechanism was effective.

# References


[1] Jennifer A. Adams, Ulf H. Danielsson, Dario Grasso, and Hector Rubinstein, *Distortion of the acoustic peaks in the CMBR due to a primordial magnetic field*, Phys.Lett. **B388** (1996), 253–258. 69

[2] Jarkko Ahonen and Kari Enqvist, *Magnetic field generation in first order phase transition bubble collisions*, Phys.Rev. **D57** (1998), 664–673. 4, 74

[3] Laila Alabidi and David Lyth, *Curvature perturbation from symmetry breaking the end of inflation*, JCAP **0608** (2006), 006. 3, 33

[4] Andreas Albrecht and Paul J. Steinhardt, *Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking*, Phys.Rev.Lett. **48** (1982), 1220–1223. 7

[5] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, *The order of the quantum chromodynamics transition predicted by the standard model of particle physics*, Nature **443** (2006), 675–678. 4, 76

[6] P.P. Avelino and E.P.S. Shellard, *Dynamical friction on cosmic string motion and magnetic field generation*, Phys.Rev. **D51** (1995), 5946–5949. 4, 76

[7] Kazuharu Bamba and Misao Sasaki, *Large-scale magnetic fields in the inflationary universe*, JCAP **0702** (2007), 030. 5, 82, 84

[8] James M. Bardeen, Paul J. Steinhardt, and Michael S. Turner, *Spontaneous Creation of Almost Scale - Free Density Perturbations in an Inflationary Universe*, Phys.Rev. **D28** (1983), 679. 23, 44

[9] Neil Barnaby and James M. Cline, *Nongaussian and nonscale-invariant perturbations from tachyonic preheating in hybrid inflation*, Phys.Rev. **D73** (2006), 106012. 3, 33

[10] ______, *Nongaussianity from Tachyonic Preheating in Hybrid Inflation*, Phys.Rev. **D75** (2007), 086004. 3, 33

[11] John D. Barrow, Pedro G. Ferreira, and Joseph Silk, *Constraints on a primordial magnetic field*, Phys.Rev.Lett. **78** (1997), 3610–3613. 67

[12] N. Bartolo, E. Komatsu, Sabino Matarrese, and A. Riotto, *Non-Gaussianity from inflation: Theory and observations*, Phys.Rept. **402** (2004), 103–266. 34

[13] N. Bartolo, S. Matarrese, and A. Riotto, *On nonGaussianity in the curvaton scenario*, Phys.Rev. **D69** (2004), 043503. 33

[14] Nicola Bartolo, Sabino Matarrese, and Antonio Riotto, *CMB Anisotropies at Second Order I*, JCAP **0606** (2006), 024. 31

[15] ______, *CMB Anisotropies at Second-Order II: Analytical Approach*, JCAP **0701** (2007), 019. 31

[16] Gordon Baym, Dietrich Bodeker, and Larry D. McLerran, *Magnetic fields produced by phase transition bubbles in the electroweak phase transition*, Phys.Rev. **D53** (1996), 662–667. 4, 73

[17] Martin L. Bernet, Francesco Miniati, Simon J. Lilly, Philipp P. Kronberg, and Miroslava Dessauges-Zavadsky, *Strong magnetic fields in normal galaxies at high redshifts*, Nature **454** (2008), 302–304. 3, 77

[18] N.D. Birrell and P.C.W. Davies, *QUANTUM FIELDS IN CURVED SPACE*, (1982). 12

[19] J.R. Bond and G. Efstathiou, *The statistics of cosmic background radiation fluctuations*, Mon.Not.Roy.Astron.Soc. **226** (1987), 655–687. 24

[20] Axel Brandenburg and Kandaswamy Subramanian, *Astrophysical magnetic fields and nonlinear dynamo theory*, Phys.Rept. **417** (2005), 1–209. 77

[21] Bao-lian Cheng and Angela V. Olinto, *Primordial magnetic fields generated in the quark - hadron transition*, Phys.Rev. **D50** (1994), 2421–2424. 4, 75

[22] Krzysztof T. Chyzy and Rainer Beck, *Magnetic fields in merging spirals - the Antennae*, Astron. Astrophys. **417** (2004), 541–555. 65

[23] Paolo Creminelli and Matias Zaldarriaga, *Single field consistency relation for the 3-point function*, JCAP **0410** (2004), 006. 27, 28

[24] Anne-Christine Davis, Matthew Lilley, and Ola Tornkvist, *Relaxing the bounds on primordial magnetic seed fields*, Phys.Rev. **D60** (1999), 021301. 72

[25] Stephen C. Davis and Marieke Postma, *SUGRA chaotic inflation and moduli stabilisation*, JCAP **0803** (2008), 015. 43

[26] Vittoria Demozzi, Andrei Linde, and Viatcheslav Mukhanov, *Supercurvaton*, JCAP **1104** (2011), 013. 5, 42

[27] Vittoria Demozzi, Viatcheslav Mukhanov, and Hector Rubinstein, *Magnetic fields from inflation?*, JCAP **0908** (2009), 025. 6, 84

[28] Andres Diaz-Gil, Juan Garcia-Bellido, Margarita Garcia Perez, and Antonio Gonzalez-Arroyo, *Magnetic field production during preheating at the electroweak scale*, Phys. Rev. Lett. **100** (2008), 241301. 4, 74

[29] Andres Diaz-Gil, Juan Garcia-Bellido, Margarita Garcia Perez, and Antonio Gonzalez-Arroyo, *Primordial magnetic fields from preheating at the electroweak scale*, JHEP **07** (2008), 043. 4, 74

[30] Konstantinos Dimopoulos, George Lazarides, David Lyth, and Roberto Ruiz de Austri, *Curvaton dynamics*, Phys.Rev. **D68** (2003), 123515. 34

[31] K. Dolag, M. Kachelriess, S. Ostapchenko, and R. Tomas, *Lower limit on the strength and filling factor of extragalactic magnetic fields*, Astrophys.J. **727** (2011), L4. 3, 65

[32] A. Dolgov, *Breaking of conformal invariance and electromagnetic field generation in the universe*, Phys.Rev. **D48** (1993), 2499–2501. 5, 80

[33] A. Dolgov and J. Silk, *Electric charge asymmetry of the universe and magnetic field generation*, Phys.Rev. **D47** (1993), 3144–3150. 5, 80

[34] A.D. Dolgov, *Generation of magnetic fields in cosmology*, (2001), To be published in collection of papers dedicated to the 70th anniversary of S.G. Matinyan. 78

[35] Ruth Durrer, Lukas Hollenstein, and Rajeev Kumar Jain, *Can slow roll inflation induce relevant helical magnetic fields?*, JCAP **1103** (2011), 037. 83

[36] Gia Dvali, Andrei Gruzinov, and Matias Zaldarriaga, *Cosmological perturbations from inhomogeneous reheating, freezeout, and mass domination*, Phys.Rev. **D69** (2004), 083505. 3, 33

[37] Gia Dvali, Oriol Pujolas, and Michele Redi, *Consistent Lorentz Violation in Flat and Curved Space*, Phys.Rev. **D76** (2007), 044028. 85, 93

[38] Damien A. Easson and Brian A. Powell, *Optimizing future experimental probes of inflation*, Phys.Rev. **D83** (2011), 043502. 61

[39] Kari Enqvist, *Primordial magnetic fields*, Int.J.Mod.Phys. **D7** (1998), 331–350. 4, 74

[40] Kari Enqvist, Asko Jokinen, Anupam Mazumdar, Tuomas Multamaki, and Antti Vaihkonen, *Non-Gaussianity from instant and tachyonic preheating*, JCAP **0503** (2005), 010. 3, 33

[41] Kari Enqvist and Tomo Takahashi, *Signatures of Non-Gaussianity in the Curvaton Model*, JCAP **0809** (2008), 012. 51

[42] Kari Enqvist and Antti Vaihkonen, *Non-Gaussian perturbations in hybrid inflation*, JCAP **0409** (2004), 006. 3, 33

[43] Torsten A. Ensslin, Corina Vogt, and Christoph Pfrommer, *Magnetic Fields in Clusters of Galaxies*, (2005). 3, 65

[44] Warren Essey, Shin'ichiro Ando, and Alexander Kusenko, *Determination of intergalactic magnetic fields from gamma ray data*, (2010). 65

[45] Jonathan Ganc and Eiichiro Komatsu, *A new method for calculating the primordial bispectrum in the squeezed limit*, JCAP **1012** (2010), 009. 27

[46] Juan Garcia-Bellido, Andrei D. Linde, and Dmitri A. Linde, *Fluctuations of the gravitational constant in the inflationary Brans-Dicke cosmology*, Phys.Rev. **D50** (1994), 730–750. 59

[47] Jaume Garriga and Alexander Vilenkin, *Anthropic prediction for Lambda and the Q catastrophe*, Prog.Theor.Phys.Suppl. **163** (2006), 245–257. 59

[48] M. Gasperini, Massimo Giovannini, and G. Veneziano, *Primordial magnetic fields from string cosmology*, Phys.Rev.Lett. **75** (1995), 3796–3799. 5, 82, 84

[49] Dario Grasso and Antonio Riotto, *On the nature of the magnetic fields generated during the electroweak phase transition*, Phys.Lett. **B418** (1998), 258–265. 4, 74

[50] Dario Grasso and Hector R. Rubinstein, *Revisiting nucleosynthesis constraints on primordial magnetic fields*, Phys.Lett. **B379** (1996), 73–79. 70

[51] ———, *Magnetic fields in the early universe*, Phys.Rept. **348** (2001), 163–266. 4, 63, 66, 70, 71, 77, 78

[52] Alan H. Guth, *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, Phys.Rev. **D23** (1981), 347–356. 7

[53] Alan H. Guth and S.Y. Pi, *Fluctuations in the New Inflationary Universe*, Phys.Rev.Lett. **49** (1982), 1110–1113. 44

[54] E. R. Harrison, *Generation of magnetic fields in the radiation era*, Mon. Not. R. Astr. Soc. **147** (1969), 279–286. 72

[55] ______, *Origin of magnetic fields in the early Universe*, Phys. Rev. Lett. **30** (1973), 188. 72

[56] S.W. Hawking, *The Development of Irregularities in a Single Bubble Inflationary Universe*, Phys.Lett. **B115** (1982), 295, Revised version. 44

[57] Burak Himmetoglu, Carlo R. Contaldi, and Marco Peloso, *Instability of anisotropic cosmological solutions supported by vector fields*, Phys.Rev.Lett. **102** (2009), 111301. 85

[58] C. Hogan, *Magnetohydrodynamic effects of a first-order cosmological phase transition*, Phys. Rev. Lett. **51** (1983), 1488. 4, 73

[59] Qing-Guo Huang, *The Trispectrum in the Multi-brid Inflation*, JCAP **0905** (2009), 005. 3, 33

[60] Asko Jokinen and Anupam Mazumdar, *Very large primordial non-gaussianity from multi-field: application to massless preheating*, JCAP **0604** (2006), 003. 3, 33

[61] K. Kajantie and H. Kurki-Suonio, *Bubble Growth and Droplet Decay in the Quark Hadron Phase Transition in the Early Universe*, Phys.Rev. **D34** (1986), 1719–1738. 75

[62] K. Kajantie, M. Laine, K. Rummukainen, and Mikhail E. Shaposhnikov, *Is there a hot electroweak phase transition at m(H) larger or equal to m(W)?*, Phys.Rev.Lett. **77** (1996), 2887–2890. 75

[63] Renata Kallosh and Andrei Linde, *New models of chaotic inflation in supergravity*, JCAP **1011** (2010), 011. 44, 45, 60, 61

[64] Renata Kallosh, Andrei Linde, and Tomas Rube, *General inflaton potentials in supergravity*, Phys.Rev. **D83** (2011), 043507. 44, 45, 60, 61

[65] Alejandra Kandus, Kerstin E. Kunze, and Christos G. Tsagas, *Primordial magnetogenesis*, (2010). 63, 73

[66] M. Kawasaki, Masahide Yamaguchi, and T. Yanagida, *Natural chaotic inflation in supergravity*, Phys.Rev.Lett. **85** (2000), 3572–3575. 3, 40, 43, 60

[67] Peter J. Kernan, Glenn D. Starkman, and Tammay Vachaspati, *Big bang nucleosynthesis constraints on primordial magnetic fields*, Phys.Rev. **D54** (1996), 7207–7214. 70

[68] E. Komatsu et al., *Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation*, Astrophys.J.Suppl. **192** (2011), 18. 1, 2, 22, 29, 33, 99

[69] Eiichiro Komatsu, *The pursuit of non-gaussian fluctuations in the cosmic microwave background*, (2002). 12, 22, 25

[70] ———, *Hunting for Primordial Non-Gaussianity in the Cosmic Microwave Background*, Class.Quant.Grav. **27** (2010), 124010. 29

[71] P. P. Kronberg et al., *A Global Probe of Cosmic Magnetic Fields to High Redshifts*, Astrophys. J. **676** (2008), 7079. 3, 77

[72] D. Larson, J. Dunkley, G. Hinshaw, E. Komatsu, M.R. Nolta, et al., *Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Power Spectra and WMAP-Derived Parameters*, Astrophys.J.Suppl. **192** (2011), 16. 70

[73] D. Lemoine and M. Lemoine, *Primordial magnetic fields in string cosmology*, Phys.Rev. **D52** (1995), 1955–1962. 5, 82

[74] Andrew R. Liddle and D.H. Lyth, *Cosmological inflation and large scale structure*, (2000). 2

[75] Andrei D. Linde, *A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems*, Phys.Lett. **B108** (1982), 389–393. 7

[76] ———, *Chaotic Inflation*, Phys.Lett. **B129** (1983), 177–181. 11, 40, 43, 44

[77] ———, *Eternally Existing Selfreproducing Chaotic Inflationary Universe*, Phys.Lett. **B175** (1986), 395–400. 44

[78] ———, *Particle physics and inflationary cosmology*, (2005). 44

[79] ———, *Inflationary Cosmology*, Lect.Notes Phys. **738** (2008), 1–54. 44

[80] Andrei D. Linde and Viatcheslav Mukhanov, *The curvaton web*, JCAP **0604** (2006), 009. 34, 44, 46, 48, 56, 59, 60, 61

[81] Andrei D. Linde and Viatcheslav F. Mukhanov, *Nongaussian isocurvature perturbations from inflation*, Phys.Rev. **D56** (1997), 535–539. 2, 34

[82] David H. Lyth, *Non-gaussianity and cosmic uncertainty in curvaton-type models*, JCAP **0606** (2006), 015. 59

[83] David H. Lyth, Carlo Ungarelli, and David Wands, *The Primordial density perturbation in the curvaton scenario*, Phys.Rev. **D67** (2003), 023503. 33, 34, 46, 51

[84] David H. Lyth and David Wands, *Generating the curvature perturbation without an inflaton*, Phys.Lett. **B524** (2002), 5–14. 2, 33

[85] David H Lyth and David Wands, *The CDM isocurvature perturbation in the curvaton scenario*, Phys.Rev. **D68** (2003), 103516. 34

[86] Juan Martin Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, JHEP **0305** (2003), 013. 27

[87] Karim A. Malik and David H. Lyth, *A numerical study of non-gaussianity in the curvaton scenario*, JCAP **0609** (2006), 008. 33

[88] Karim A. Malik, David Wands, and Carlo Ungarelli, *Large scale curvature and entropy perturbations for multiple interacting fluids*, Phys.Rev. **D67** (2003), 063516. 37

[89] Jerome Martin and Jun'ichi Yokoyama, *Generation of Large-Scale Magnetic Fields in Single-Field Inflation*, JCAP **0801** (2008), 025. 5, 83

[90] J.J. Matese and R.F. O'Connell, *Neutron Beta Decay in a Uniform Constant Magnetic Field*, Phys.Rev. **180** (1969), 1289–1292. 69

[91] Takeo Moroi and Tomo Takahashi, *Effects of cosmological moduli fields on cosmic microwave background*, Phys.Lett. **B522** (2001), 215–221. 34

[92] ———, *Non-Gaussianity and Baryonic Isocurvature Fluctuations in the Curvaton Scenario*, Phys.Lett. **B671** (2009), 339–344. 34

[93] V. Mukhanov, *Physical foundations of cosmology*, (2005). 2, 9, 16, 23, 44, 86, 89

[94] Viatcheslav Mukhanov and Sergei Winitzki, *Introduction to quantum effects in gravity*, (2007). 12, 86

[95] Viatcheslav F. Mukhanov, *Gravitational Instability of the Universe Filled with a Scalar Field*, JETP Lett. **41** (1985), 493–496. 44

[96] Viatcheslav F. Mukhanov and G.V. Chibisov, *Quantum Fluctuation and Nonsingular Universe. (In Russian)*, JETP Lett. **33** (1981), 532–535. 44

[97] ———, *The Vacuum energy and large scale structure of the universe*, Sov.Phys.JETP **56** (1982), 258–265. 7

[98] Viatcheslav F. Mukhanov, H.A. Feldman, and Robert H. Brandenberger, *Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions*, Phys.Rept. **215** (1992), 203–333. 15, 23

[99] Atsushi Naruko and Misao Sasaki, *Large non-Gaussianity from multi-brid inflation*, Prog.Theor.Phys. **121** (2009), 193–210. 3, 33

[100] A. Neronov and I. Vovk, *Evidence for strong extragalactic magnetic fields from Fermi observations of TeV blazars*, Science **328** (2010), 73–75. 3, 65

[101] Daisuke Nitta, Eiichiro Komatsu, Nicola Bartolo, Sabino Matarrese, and Antonio Riotto, *CMB anisotropies at second order III: bispectrum from products of the first-order perturbations*, JCAP **0905** (2009), 014. 30

[102] L. Parker, *Particle creation in expanding universes*, Phys.Rev.Lett. **21** (1968), 562–564. 4

[103] Cyril Pitrou, Jean-Philippe Uzan, and Francis Bernardeau, *Cosmic microwave background bispectrum on small angular scales*, Phys.Rev. **D78** (2008), 063526. 31

[104] ______, *The cosmic microwave background bispectrum from the non-linear evolution of the cosmological perturbations*, JCAP **1007** (2010), 003. 31

[105] Ted Pyne and Sean M. Carroll, *Higher order gravitational perturbations of the cosmic microwave background*, Phys.Rev. **D53** (1996), 2920–2929. 25

[106] Jean M. Quashnock, Abraham Loeb, and David N. Spergel, *Magnetic Field Generation During the Cosmological QCD Phase Transition*, (1988). 4, 75

[107] Bharat Ratra, *Cosmological 'seed' magnetic field from inflation*, Astrophys.J. **391** (1992), L1–L4. 5, 82, 84

[108] Bharat Ratra and P.J.E. Peebles, *Inflation in an open universe*, Phys.Rev. **D52** (1995), 1837–1894. 82

[109] M. J. Rees, *The origin and cosmogonic implications of seed magnetic fields.* 72

[110] Beth A. Reid et al., *Baryon Acoustic Oscillations in the Sloan Digital Sky Survey Data Release 7 Galaxy Sample*, Mon.Not.Roy.Astron.Soc. **401** (2010), 2148–2168. 22, 33

[111] Adam G. Riess, Lucas Macri, Stefano Casertano, Megan Sosey, Hubert Lampeitl, et al., *A Redetermination of the Hubble Constant with the Hubble Space Telescope from a Differential Distance Ladder*, Astrophys.J. **699** (2009), 539–563. 22, 33

[112] D.S. Salopek and J.R. Bond, *Nonlinear evolution of long wavelength metric fluctuations in inflationary models*, Phys.Rev. **D42** (1990), 3936–3962. 25

[113] Misao Sasaki, Jussi Valiviita, and David Wands, *Non-Gaussianity of the primordial perturbation in the curvaton model*, Phys.Rev. **D74** (2006), 103003. 51

[114] Alexei A. Starobinsky, *Relict Gravitation Radiation Spectrum and Initial State of the Universe. (In Russian)*, JETP Lett. **30** (1979), 682–685. 7

[115] ______, *Dynamics of Phase Transition in the New Inflationary Universe Scenario and Generation of Perturbations*, Phys.Lett. **B117** (1982), 175–178. 44

[116] F. Tavecchio, G. Ghisellini, L. Foschini, G. Bonnoli, G. Ghirlanda, et al., *The intergalactic magnetic field constrained by Fermi/LAT observations of the TeV blazar 1ES 0229+200*, Mon.Not.Roy.Astron.Soc. **406** (2010), L70–L74. 3, 65

[117] Max Tegmark and Martin J. Rees, *Why is the CMB fluctuation level 10**(-5)?*, Astrophys.J. **499** (1998), 526–532. 59

[118] Ola Tornkvist, Anne-Christine Davis, Konstantinos Dimopoulos, and Tomislav Prokopec, *Large scale primordial magnetic fields from inflation and preheating*, (2000), 443–446. 5, 81

[119] Michael S. Turner and Lawrence M. Widrow, *Inflation Produced, Large Scale Magnetic Fields*, Phys.Rev. **D37** (1988), 2743. 4, 5, 78, 79, 85

[120] T. Vachaspati, *Magnetic fields from cosmological phase transitions*, Phys.Lett. **B265** (1991), 258–261. 4, 73

[121] Tanmay Vachaspati and Alexander Vilenkin, *Large scale structure from wiggly cosmic strings*, Phys.Rev.Lett. **67** (1991), 1057–1061. 4, 76

[122] A. Vilenkin, *Parity nonconservation and rotating black holes*, Phys. Rev. Lett. **41** (1978), 1575. 72

[123] A. Vilenkin and A. D. Leahy, *Parity nonconservation and the origin of cosmic magnetic fields*, Astrophys. J. **254** (1982), 77. 72

[124] Corina Vogt and Torsten A. Ensslin, *Measuring the Cluster Magnetic Field Power Spectra from Faraday Rotation Maps of Abell 400, Abell 2634 and Hydra A*, Astron. Astrophys. **412** (2003), 373–386. 65

[125] J. Wess and J. Bagger, *Supersymmetry and supergravity*, (1992). 40

[126] Lawrence M. Widrow, *Origin of galactic and extragalactic magnetic fields*, Rev.Mod.Phys. **74** (2002), 775–823. 63

[127] Edward Witten, *Superconducting Strings*, Nucl.Phys. **B249** (1985), 557–592. 4, 76

[128] Masahide Yamaguchi, *Supergravity based inflation models: a review*, Class.Quant.Grav. **28** (2011), 103001. 3

[129] Matias Zaldarriaga, *Non-Gaussianities in models with a varying inflaton decay rate*, Phys.Rev. **D69** (2004), 043508. 3, 33

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